# Toric Fiber Products in Geometric Modeling


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## Geometric Modeling

In Geometric Modeling, one is concerned with describing shapes in Euclidean space. Particularly, one is interested in finding nice parametrizations of such shapes. Such parametrizations are defined by a set of blending functions $\beta_i$, a set of control points $\mathbf{p}_i \in \mathbb{R}^d$ and weights $w_i \in \mathbb{R}_{>0}$ giving rise to a Bézier patch

$$\Phi(\mathbf{t}) = \sum_i w_i \beta_i(\mathbf{t})\mathbf{p}_i.$$

In our case, the $\mathbf{p}_i$ are the lattice points of a polytope $P \subseteq \mathbb{R}^d$ and the $\beta_i$ are functions on $P$. It is desirable that the weighted polytope $(P, w)$ satisfies the property of rational linear precision.

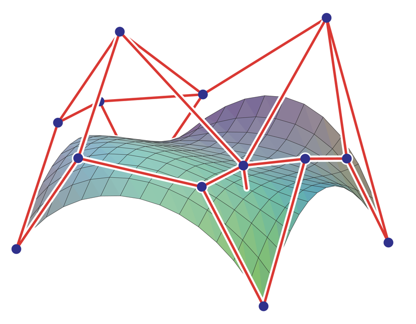

Fig. 1: A Bézier patch with control points, taken from [2]

## Rational Linear Precision (RLP)

- Let $\mathcal{B}$ be an integer point configuration in $\mathbb{Z}^d$ and $P = \mathrm{Conv}(\mathcal{B})$ the associated polytope
- Let $X_{\mathcal{B},w}$ be the scaled projective toric variety defined by $P$ and weights $w$

**Definition.** The pair $(P, w)$ has RLP if there is a set of rational functions $\{\hat\beta_{\mathbf{b}}\}_{\mathbf{b}\in\mathcal{B}}$ on $\mathbb{C}^d$ satisfying:

1. $\sum_{\mathbf{b}\in\mathcal{B}} \hat\beta_{\mathbf{b}} = 1$.
2. The functions $\{\hat\beta_{\mathbf{b}}\}_{\mathbf{b}\in\mathcal{B}}$ define a rational parametrization
$$\hat\beta : \mathbb{C}^d \dashrightarrow X_{\mathcal{B},w} \subset \mathbb{P}^{|\mathcal{B}|-1}, \quad \hat\beta(\mathbf{t}) = (\hat\beta_{\mathbf{b}}(\mathbf{t}))_{\mathbf{b}\in\mathcal{B}}.$$
3. For every $\mathbf{p} \in \mathrm{Relint}(P) \subset \mathbb{C}^d$, $\hat\beta_{\mathbf{b}}(\mathbf{p})$ is defined and is a nonnegative real number.
4. Linear precision: $\sum_{\mathbf{b}\in\mathcal{B}} \hat\beta_{\mathbf{b}}(\mathbf{p})\mathbf{b} = \mathbf{p}$ for all $\mathbf{p} \in P$.

Connection to Algebraic Statistics: $(P, w)$ has RLP iff the scaled toric variety $X_{\mathcal{B},w}$ has MLdegree one

## Toric Fiber Products

- Fix two integral point configurations $\mathcal{B} = \{\mathbf{b}^i_j : i \in [r], j \in [s_i]\} \subseteq \mathbb{Z}^{d_1}$ and $\mathcal{C} = \{\mathbf{c}^i_k : i \in [r], k \in [t_i]\} \subseteq \mathbb{Z}^{d_2}$
- Equip them with a multigrading $\mathcal{A} = \{\mathbf{a}^i : i \in [r]\} \subseteq \mathbb{Z}^d$ such that $\deg(\mathbf{b}^i_j) = \deg(\mathbf{c}^i_k) = \mathbf{a}^i$ is linear

**Definition.** The toric fiber product of $\mathcal{B}$ and $\mathcal{C}$ is the point configuration $\mathcal{B} \times_{\mathcal{A}} \mathcal{C}$ given by

$$\mathcal{B} \times_{\mathcal{A}} \mathcal{C} = \{(\mathbf{b}^i_j, \mathbf{c}^i_k) : \mathbf{b}^i_j \in \mathcal{B},\ \mathbf{c}^i_k \in \mathcal{C},\ \deg(\mathbf{b}^i_j) = \deg(\mathbf{c}^i_k)\}.$$

$$\begin{array}{cccc} \mathbf{b}^1_1 & \mathbf{b}^1_2 & \mathbf{b}^2_1 & \mathbf{b}^2_2 \end{array} \quad \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\begin{array}{ccccc} \mathbf{c}^1_1 & \mathbf{c}^1_2 & \mathbf{c}^1_3 & \mathbf{c}^2_1 & \mathbf{c}^2_2 \end{array} \quad \begin{pmatrix} 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\begin{array}{cccccccccc} \mathbf{b}^1_1 & \mathbf{b}^1_1 & \mathbf{b}^1_1 & \mathbf{b}^1_2 & \mathbf{b}^1_2 & \mathbf{b}^1_2 & \mathbf{b}^2_1 & \mathbf{b}^2_1 & \mathbf{b}^2_2 & \mathbf{b}^2_2 \\ \mathbf{c}^1_1 & \mathbf{c}^1_2 & \mathbf{c}^1_3 & \mathbf{c}^1_1 & \mathbf{c}^1_2 & \mathbf{c}^1_3 & \mathbf{c}^2_1 & \mathbf{c}^2_2 & \mathbf{c}^2_1 & \mathbf{c}^2_2 \end{array} \quad \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 0 & 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

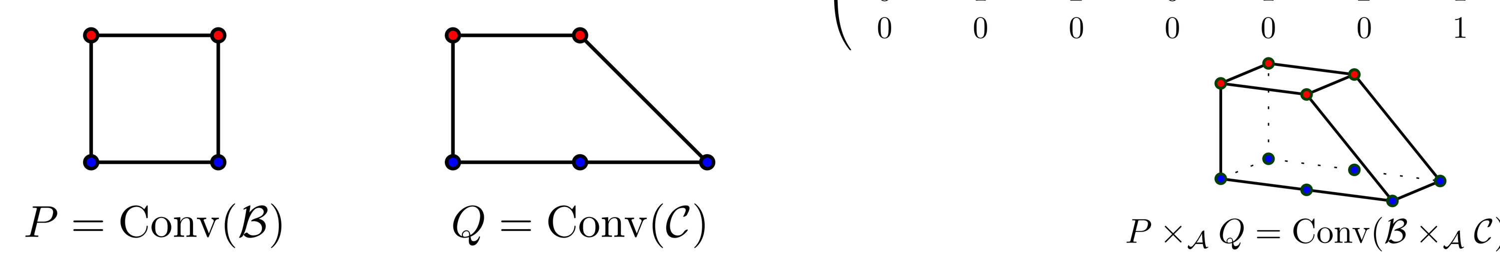


Fig. 2: Illustration of the toric fiber product $\mathcal{B} \times_{\mathcal{A}} \mathcal{C}$ with $\mathcal{A} = \{e_1, e_2\}$; the multigrading corresponds to the coloring of the lattice points

## Motivation

- It was known that the MLdegree is multiplicative with respect to toric fiber products [4]
- Therefore, toric fiber products preserve RLP

**Question.** What do the blending functions of a toric fiber product that satisfy RLP look like?

## Proof Idea

- To show that the two expressions in (★) agree we use a torus action induced by the multigrading:
$$T_{\mathcal{A}} \times X_{\mathcal{B}\times_{\mathcal{A}}\mathcal{C}} \to X_{\mathcal{B}\times_{\mathcal{A}}\mathcal{C}}$$
$$(t^1, \dots, t^{|\mathcal{A}|}).\left(x^i_{j,k}\right)^{i\in|\mathcal{A}|}_{(j,k)\in|\mathcal{B}^i\times\mathcal{C}^i|} = \left(t^i x^i_{j,k}\right)^{i\in|\mathcal{A}|}_{(j,k)\in|\mathcal{B}^i\times\mathcal{C}^i|}$$
- The two expressions always lie in the same $T_{\mathcal{A}}$-orbit
- On every orbit, we can find a point on which they agree; on the orbit of maximal dimension, this is the point where they evaluate to an MLE
- The rest of the proof amounts to checking the four properties of RLP

## Summary


- In this work, we introduced toric fiber products, well-known in Algebraic Statistics, to the context of Geometric Modeling
- This allows for the construction of new polytopes having the property of rational linear precision from lower dimensional ones, in analogy to constructing statistical models with rational MLE
- Particularly, we can give an explicit description of blending functions satisfying RLP for the toric fiber product polytope
- In our paper, we also give an explicit description of the Horn matrix of a toric fiber product


## Main Result

- We assume $\mathcal{A}$ to be linearly independent
- For weight vectors $w, \tilde w$ of $\mathcal{B}, \mathcal{C}$, define toric fiber product weights $w_{\mathcal{B}\times_{\mathcal{A}}\mathcal{C}} = (w^i_j \tilde w^i_k)^{i\in|\mathcal{A}|}_{(j,k)\in|\mathcal{B}^i\times\mathcal{C}^i|}$

**Theorem.** If $P$ and $Q$ are polytopes with rational linear precision for weights $w, \tilde w$, respectively, then the toric fiber product $P \times_{\mathcal{A}} Q$ has rational linear precision with vector of weights $w_{\mathcal{B}\times_{\mathcal{A}}\mathcal{C}}$. Moreover, blending functions with rational linear precision for $P \times_{\mathcal{A}} Q$ are given by

$$\beta^i_{j,k}(\mathbf{p}, \mathbf{q}) = \frac{\beta^i_j(\mathbf{p})\beta^i_k(\mathbf{q})}{\sum_{j'\in|\mathcal{B}^i|}\beta^i_{j'}(\mathbf{p})} = \frac{\beta^i_j(\mathbf{p})\beta^i_k(\mathbf{q})}{\sum_{k'\in|\mathcal{C}^i|}\beta^i_{k'}(\mathbf{q})} \qquad (★)$$

where $(\mathbf{p}, \mathbf{q}) \in P \times_{\mathcal{A}} Q$.

- It turns out that the two expressions above are equal on $P \times_{\mathcal{A}} Q$; this is the hardest part of the proof

## Example

- The square (Fig. 2 left) has RLP with weights $(1,1,1,1)$ and blending functions
$$\beta_{\binom{0}{0}} = (1\text{-}x_1)(1\text{-}x_2), \qquad \beta_{\binom{1}{0}} = x_2(1\text{-}x_1), \qquad \beta_{\binom{0}{1}} = x_1(1\text{-}x_2), \qquad \beta_{\binom{1}{1}} = x_1 x_2$$
- The trapezoid (Fig. 2 center) has RLP with weights $(1,2,1,1,1)$ and blending functions
$$\tilde\beta_{\binom{0}{0}} = \frac{(1\text{-}y_2)(2\text{-}y_1\text{-}y_2)^2}{(2\text{-}y_2)^2}, \quad \tilde\beta_{\binom{1}{0}} = \frac{2y_1(1\text{-}y_2)(2\text{-}y_1\text{-}y_2)}{(2\text{-}y_2)^2}, \quad \tilde\beta_{\binom{2}{0}} = \frac{y_1^2(1\text{-}y_2)}{(2\text{-}y_2)^2}, \quad \tilde\beta_{\binom{0}{1}} = \frac{y_2(2\text{-}y_1\text{-}y_2)}{2\text{-}y_2}, \quad \tilde\beta_{\binom{1}{1}} = \frac{y_1y_2}{2\text{-}y_2}$$
- Their toric fiber product (Fig. 2 right) has RLP with weights $(1,2,1,1,2,1,1,1,1,1)$. One blending function is
$$\beta^1_{2,3} = \frac{\beta^1_2\tilde\beta^1_3}{\beta^1_1 + \beta^2_1} = \frac{x_1(1\text{-}x_2)y_1^2(1\text{-}y_2)}{1\text{-}x_2} = \frac{\beta^1_2\tilde\beta^1_3}{\tilde\beta^1_1 + \tilde\beta^1_2 + \tilde\beta^1_3} = \frac{x_1(1\text{-}x_2)y_1^2(1\text{-}y_2)}{1\text{-}y_2}$$
- While the denominators are not the same, the two expressions above are equal on $\mathrm{Relint}(P \times_{\mathcal{A}} Q)$

## References


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