MIRROR SYMMETRY FOR $\mathbb{C}^*$

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Contents

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1. Introduction

Mirror symmetry first arouse from string theory: in the late 1980s physicists discovered that two different Calabi–Yau compactifications of extra dimensions can lead to the same physics. There are string theories of type IIA and of type IIB and a Calabi–Yau manifold in the A-model corresponds to a "mirror dual" Calabi–Yau manifold in the B-model resulting in the same physics, though the manifolds can have very different structures: on the A-side the manifold has a symplectic structure, whereas the dual on the B-side has a complex-algebraic structure. This phenomenon became interesting for mathematicians when in 1991 Candelas et al. [13] applied mirror symmetry to enumerative geometry calculating the number of cubic curves on a quintic threefold (317206375) and thereby correcting (and extending) an earlier result by Ellingsrud and Strømme who could later confirm Candelas' number using classical techniques from algebraic geometry [16]. Since then it is a big mathematical challenge to formulate and prove mirror symmetry in a mathematically rigorous way.

One such attempt is the homological mirror symmetry conjecture by Kontsevich, presented on the ICM in 1994 [25].

Conjecture (Homological Mirror Symmetry)**.** *Let (M, ω) be a $2n$-dimensional symplectic manifold with $c_1(M) = 0$. Then the derived Fukaya category $DF(M)$ (or some suitable enlargement) is equivalent to $D^b\mathrm{Coh}(\check{M})$ for an n-dimensional complex algebraic variety $\check{M}$.*

The conjecture has been proved in some cases. In 1998 Polishchuk and Zaslow [33] gave a proof for the case of an elliptic curve (which is mirror dual to its dual curve). The proof was extended to Abelian varieties by Fukaya in 2002 [18]. Seidel gave a proof for K3 surfaces (on the symplectic side) in 2003 [36].

Later the conjecture has been extended to non-Calabi–Yau manifolds by considering Landau–Ginzburg models, consisting of a variety X together with a holomorphic function $X \to \mathbb{C}$ called superpotential [23]. In this extension the conjecture has been proved for example for curves of genus $g \geq 2$ [35], [15]. Moreover, there are results for toric Fano varieties, confer [2],[1]. [1] It might be worth noting that mirror symmetry is really a "mirror symmetric" statement interchanging the A- with the B-model in the following sense: the A-model on a Landau–Ginzburg model (given by the derived Fukaya category) is equivalent to the B-model on the Fano variety (given by the derived category of coherent sheaves); vice versa, the B-model on the Landau–Ginzburg model (given by the category of matrix factorizations) is equivalent to the A-model on the Fano variety (given by the derived Fukaya category after equipping the variety with a Kähler form) [31].

The easiest non-Calabi–Yau case is $\mathbb{C}^*$ being mirror dual to itself; here we consider $\mathbb{C}^*$ as T^*S^1 on the symplectic side and as $\mathrm{Spec}(\mathbb{C}[x, x^{-1}])$ on the algebro-geometric side. As $\mathbb{C}^*$ is non-compact, we have to deal with the *wrapped* Fukaya category. Homological mirror symmetry is then the following statement.

[1] This should by no means be a complete list of all known results!

Theorem (Homological Mirror Symmetry for $\mathbb{C}^*$)**.** *There is an equivalence of triangulated categories between the derived wrapped Fukaya category of T^*S^1 and the derived category of coherent sheaves over* $\mathrm{Spec}(\mathbb{C}[x, x^{-1}])$*,*

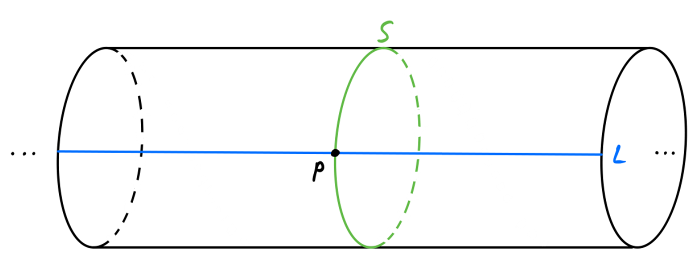
$$D\mathcal{W}(T^*S^1) \cong D^b\mathrm{Coh}(\mathrm{Spec}(\mathbb{C}\left[x, x^{-1}\right])).$$

This essay's goal is to first introduce the concept of wrapped Fukaya categories and then to present a proof this theorem.

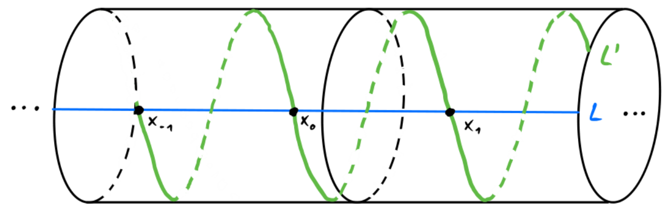
The wrapped Fukaya category of a cotangent bundle was first introduced by Fukaya, Seidel and Smith in 2008 [19]. In this paper the authors also conjecture a description of the wrapped Fukaya category of a cotangent bundle which contains the theorem above as a special case. A general definition of wrapped Fukaya categories was given by Abouzaid and Seidel in 2010 [6]. Homological mirror symmetry for $\mathbb{C}^*$ was first rigorously proven within a more general setting by Abouzaid when he described the wrapped Floer cohomology of a cotangent fibre in terms of some loop space [5] and showed that a cotangent fibre generates the wrapped Fukaya category [4]. In this essay we will mainly follow the route as in Auroux's introductory paper [8].

What is the wrapped Fukaya category $\mathcal{W}$? First of all, it is *not* a category in the usual sense, but an A_∞-category. It may lack the existence of identity morphisms and the composition of morphisms is not necessarily associative. Instead, an A_∞-category possesses certain higher product operations on the morphism spaces, in particular a differential and a product which is associative up to homotopy. We introduce A_∞-categories in §2.1. As homological mirror symmetry is a statement about the equivalence of triangulated categories, it is necessary to turn $\mathcal{W}$ into a triangulated category by passing to its derived version $D\mathcal{W}$. This is done by first enlarging $\mathcal{W}$ to its twisted category making it triangulated and then taking zeroth cohomology of the morphism spaces, as is explained in §2.2.

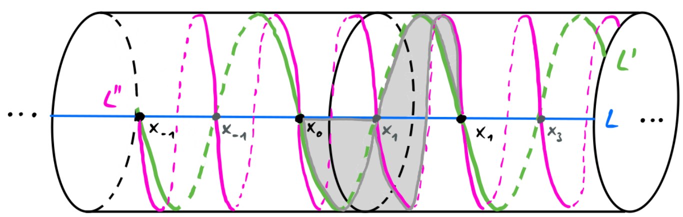
The formal definition of the (wrapped) Fukaya category of a symplectic manifold M is quite technical; however, the general idea is quickly sketched. The objects are (certain) Lagrangian submanifolds of M and the morphism space between two such Lagrangians L_0 and L_1 is given by the *(wrapped) Floer complex* $CW(L_0, L_1)$. This complex is generated over $\mathbb{C}$ by intersection points between L_0 and L_1. In the following figure we can see two Lagrangian submanifolds of T^*S^1, a cotangent fibre L and the zero section S.



They intersect in a single point p, implying $CW(L, S) = \mathbb{C} \cdot p$. We would also like to define the Floer complex in situations where the two Lagrangians are not intersecting transversely, in particular we would like to make sense of $CW(L, L)$. To make intersections again transverse, one perturbs one of the Lagrangians by a Hamiltonian diffeomorphism. In the example of T^*S^1 above, the perturbed version L' of L then wraps around the cylinder intersecting L in infinitely many points labeled by x_i.



The points x_i satisfy certain relations among each other; the product between points x_i and x_j is determined by "triangles" (pseudo-holomorphic disks with three marked points) with two vertices being x_i and x_j and edges lying on L, L' and L'' where the latter one is also a perturbed version of L but wrapped twice as fast around the cylinder as L'. In the following there is a triangle depicted determining the product $x_0 \cdot x_1 = x_1$.



In general we will obtain the rule $x_i \cdot x_j = x_{i+j}$ and thus an isomorphism of rings $CW(L, L) \cong \mathbb{C}[x, x^{-1}]$ where x_i is mapped to x^i. This isomorphism lies at the heart of the proof of homological mirror symmetry for $\mathbb{C}^*$ which follows by using Abouzaid's result that a cotangent fibre generates $\mathcal{W}(T^*S^1)$. Under mirror symmetry, the cotangent fibre L corresponds to the structure sheaf $\mathcal{O}_{\mathbb{C}^*}$ and the zero section S to the skyscraper sheaf $\mathcal{O}_1$. The observation that $CW(L, S) \cong \mathbb{C}$ then resembles the fact that $\hom(\mathcal{O}_{\mathbb{C}^*}, \mathcal{O}_1) \cong \mathbb{C}$. A detailed description of $\mathcal{W}(T^*S^1)$ and the proof of our main theorem as well as further correspondences under mirror symmetry are given in §5 of this essay.

To define (wrapped) Fukaya categories (which is done in §4) it is first necessary to introduce Floer cohomology. It was first invented by Floer in the late 1980s to prove Arnold's conjecture on the number of fixed points of a Hamiltonian diffeomorphism applied to a Lagrangian submanifold [17] and later generalised by Fukaya, Oh, Ohta and Ono [20]. The definition of Floer cohomology involves a lot of technical details and we address them in §3.

The Floer complex is generated by intersection points, as stated above. To define the Floer differential one has to count pseudo-holomorphic strips with boundary in the Lagrangians and converging to two intersection points—this is similar to the triangle seen in the example above but with only two instead of three points. We will define pseudo-holomorphic disks and the Floer differential in §3.1. A priori there does not exist a grading on the Floer complex, additional structure on the Lagrangians is needed. A detailed construction of the grading is done in §3.2, also including a definition of the Maslov index. The counting of pseudo-holomorphic disks used in the definition of the Floer differential is done by considering the zero-dimensional component of the moduli space of pseudo-holomorphic disks; the construction of this moduli space as well as a calculation of its dimension are sketched in §3.3. In §3.4 Hamiltonian perturbations are introduced guaranteeing that intersections (and self-intersections) between Lagrangians are always transverse. A proof that the Floer differential is actually a differential, i.e. that $\partial^2 = 0$, is given in §3.5, where the reader also finds some comments on why the constructions are independent of the choices we made. In §3.6 higher product operations on the Floer complex are introduced; they make the Fukaya category into an A_∞-category. The technical assumptions necessary to make Floer cohomology (and thus also the Fukaya category) well-defined are summarised in §3.7 where we also compute a first detailed example.

After proving homological mirror symmetry for $\mathbb{C}^*$ this essay is concluded by a brief overview of partially wrapped Fukaya categories leading to mirror symmetry results for $\mathbb{C}$ and $\mathbb{P}^1$.

Notation. Throughout this essay, if not otherwise mentioned, M will always denote a symplectic manifold (M, ω) of dimension $2n$. When writing varieties like $\mathbb{A}^1$ or $\mathbb{P}^1$ we always consider them as varieties over $\mathbb{C}$.

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2. A_∞-STRUCTURES

In this section the concept of A_∞-categories is briefly introduced. They build the framework for the Fukaya category. Roughly speaking, in an A_∞-category the composition of morphisms is not associative but "associative up to homotopy". For the whole section, let k denote a field. We follow the exposition in [24] and [34, §I.1] where the interested reader can also find more details.

2.1. A_∞-categories.

We start by giving the definition of an A_∞-algebra, sort of the easiest instance of an A_∞-category (it is an A_∞-category with only one object).

Definition 2.1. An *A_∞-algebra* is a $\mathbb{Z}$-graded k-vector space

$$A = \bigoplus_{i \in \mathbb{Z}} A^i$$

endowed with a family of homogeneous k-multilinear maps $\{\mu^d\}_{d \geq 1}$

$$\mu^d \colon A^{\otimes d} \to A[2-d]$$

such that the *A_∞-relations* hold

$$\sum_{\substack{l+k+m=d \\ l,m \geq 0, k>0}} (-1)^{\alpha} \mu^{d-k+1}(a_d, \ldots, a_{m+k+1}, \mu^k(a_{m+k}, \ldots, a_{m+1}), a_m, \ldots, a_1) = 0 \qquad (1)$$

for all $d \geq 1$ and $a_i \in A$, where $\alpha_m \coloneqq |a_1| + \cdots + |a_m| - m$.

In the following we will often disregard signs for the sake of brevity and clarity.

Remark 2.2. Let us take a look on the first few A_∞-relations:

(1) for $d = 1$, we get $\mu^1 \circ \mu^1 = 0$, i.e. (A, μ^1) forms a differential complex;
(2) for $d = 2$, we obtain $\mu^1(\mu^2(a_2, a_1)) = \pm\mu^2(\mu^1(a_2), a_1) \pm \mu^2(a_2, \mu^1(a_1))$ which is, when we interpret μ^1 as a differential and μ^2 as multiplication, just the Leibniz rule;
(3) for $d = 3$, we have $\mu^2(\mu^2(a_3, a_2), a_2) \pm \mu^2(a_3, \mu^2(a_2, a_1)) = \pm\mu^1(\mu^3(a_3, a_2, a_1)) \pm \mu^3(\mu^1(a_3), a_2, a_1) \pm \mu^3(a_3, \mu^1(a_2), a_1) \pm \mu^3(a_3, a_2, \mu^1(a_1))$ which means that the multiplication given by μ^2 is associative up to some "homotopy" given by μ^3.

Definition 2.3. A *non-unital A_∞-category* $\mathcal{A}$ comprises the data of a set of objects $\mathrm{Ob}(\mathcal{A})$, for any two objects $X_0, X_1 \in \mathrm{Ob}(\mathcal{A})$ a $\mathbb{Z}$-graded k-vector space $\hom_{\mathcal{A}}(X_0, X_1)$, and composition maps of every order $d \geq 1$

$$\mu^d_{\mathcal{A}} \colon \hom_{\mathcal{A}}(X_{d-1}, X_d) \otimes \ldots \otimes \hom_{\mathcal{A}}(X_0, X_1) \to \hom_{\mathcal{A}}(X_0, X_d)[2-d]$$

which satisfy the A_∞-relations (1).

Remark 2.4. Non-unital means that identity morphisms do not necessarily exist (though they might exist). Note that an A_∞-category is not a category in the usual sense.

Definition 2.5. A *non-unital A_∞-functor* $\mathcal{F}$ between two non-unital A_∞-categories $\mathcal{A}$ and $\mathcal{B}$ consists of a map $\mathcal{F}_{\mathrm{Ob}} \colon \mathrm{Ob}(\mathcal{A}) \to \mathrm{Ob}(\mathcal{B})$ and multilinear maps of every order $d \geq 1$

$$\mathcal{F}^d \colon \hom_{\mathcal{A}}(X_{d-1}, X_d) \otimes \ldots \otimes \hom_{\mathcal{A}}(X_0, X_1) \to \hom_{\mathcal{B}}(\mathcal{F}_{\mathrm{Ob}}(X_0), \mathcal{F}_{\mathrm{Ob}}(X_d))[1-d]$$

satisfying the relations

$$\sum_{r\geq 1}\sum_{s_1+\cdots+s_r=d} \mu^r_{\mathcal{B}}(\mathcal{F}^{s_r}(a_d,\ldots,a_{d-s_r+1}),\ldots,\mathcal{F}^{s_1}(a_{s_1},\ldots,a_1))$$
$$=\sum_{m,n}(-1)^{\alpha_n}\mathcal{F}^{d-m+1}(a_d,\ldots,a_{n+m+1},\mu^m_{\mathcal{A}}(a_{n+m},\ldots,a_{n+1}),a_n,\ldots,a_1).$$

2.2. **Twisted complexes and the derived category.** Mirror Symmetry is a statement about the equivalence of the Fukaya category of a space and the derived category of coherent sheaves of its mirror space. Now the former one is an A_∞-category whereas the latter one is a derived category—how can we compare them? To remedy this issue we make the Fukaya category into the *derived* Fukaya category and mirror symmetry then becomes an equivalence of triangulated categories.

Definition 2.6. Let $\mathcal{A}$ be an A_∞-category. As seen in Remark 2.2, any morphism space $\hom_{\mathcal{A}}(X_0,X_1)$ together with $\mu^1_{\mathcal{A}}$ forms a cochain complex. The *cohomology category* $H^*(\mathcal{A})$ of $\mathcal{A}$ is the (non-unital) category with objects $\mathrm{Ob}(\mathcal{A})$ and morphism spaces the cohomology groups $H^*(\hom_{\mathcal{A}}(X_0,X_1),\mu^1_{\mathcal{A}})$. The composition of morphisms is given by the formula

$$[a_2]\circ[a_1]=(-1)^{|a_1|}\left[\mu^2_{\mathcal{A}}(a_2,a_1)\right]. \tag{2}$$

We denote by $H^0(\mathcal{A})$ its subcategory with morphism spaces the degree zero cohomology groups.

It may happen that the cohomology category $H^*(\mathcal{A})$ lacks the existence of identity morphisms and is therefore not a category in the usual sense but a non-unital category (sometimes also called a semicategory). If $H^*(\mathcal{A})$ has identity morphisms for every object we say $\mathcal{A}$ is *cohomologically unital.* Let us check that in this case $H^*(\mathcal{A})$ and $H^0(\mathcal{A})$ are in fact honest categories.

Lemma 2.7. *Let $\mathcal{A}$ be a cohomologically unital A_∞-category. Then $H^*(\mathcal{A})$ and $H^0(\mathcal{A})$ are categories (in the usual sense).*

Proof. Compositions of morphisms in $H^*(\mathcal{A})$ exist and are given by (2); suppose

$$a_1\in\ker(\mu^1_{\mathcal{A}}\colon \hom_{\mathcal{A}}(X_0,X_1)\to\hom_{\mathcal{A}}(X_0,X_1)[1])$$

and

$$a_2\in\ker(\mu^1_{\mathcal{A}}\colon \hom_{\mathcal{A}}(X_1,X_2)\to\hom_{\mathcal{A}}(X_1,X_2)[1]),$$

then using the A_∞-relation we have

$$\mu^1_{\mathcal{A}}(\mu^2_{\mathcal{A}}(a_2,a_1))=\pm\mu^2_{\mathcal{A}}(\underbrace{\mu^1_{\mathcal{A}}(a_2)}_{=0},a_1)\pm\mu^2_{\mathcal{A}}(a_2,\underbrace{\mu^1_{\mathcal{A}}(a_1)}_{=0})=0$$

so $\mu^2_{\mathcal{A}}(a_2,a_1)$ represents a cohomology class and the expression (2) is well-defined.

As μ^2 has degree zero, if

$$[a_1]\in H^0(\hom_{\mathcal{A}}(X_0,X_1),\mu^1_{\mathcal{A}}) \text{ and } [a_2]\in H^0(\hom_{\mathcal{A}}(X_1,X_2),\mu^1_{\mathcal{A}})$$

then $[\mu^2_{\mathcal{A}}(a_2,a_1)]\in H^0(\hom_{\mathcal{A}}(X_0,X_2),\mu^1_{\mathcal{A}})$, so (2) also gives a well-defined composition in $H^0(\mathcal{A})$.

Identity morphisms exist by assumption that $\mathcal{A}$ is cohomologically unital. It remains to check associativity of the composition. We will do this modulo signs: suppose $[a_i] \in H^*(\hom_{\mathcal{A}}(X_{i-1}, X_i))$ for $i = 1, 2, 3$; then

$$[a_3] \circ ([a_2] \circ [a_1]) = \pm [a_3] \circ \left[\mu^2_{\mathcal{A}}(a_2, a_1)\right] = \pm \left[\mu^2_{\mathcal{A}}(a_3, \mu^2_{\mathcal{A}}(a_2, a_1))\right]$$

and

$$([a_3] \circ [a_2]) \circ [a_1] = \pm \left[\mu^2_{\mathcal{A}}(a_3, a_2)\right] \circ [a_1] = \pm \left[\mu^2_{\mathcal{A}}(\mu^2_{\mathcal{A}}(a_3, a_2), a_1)\right].$$

The A_∞-relation in degree three gives

$$\begin{aligned} \mu^2_{\mathcal{A}}(\mu^2_{\mathcal{A}}(a_3, a_2), a_2) \pm \mu^2_{\mathcal{A}}(a_3, \mu^2_{\mathcal{A}}(a_2, a_1)) = {} & \pm \mu^1_{\mathcal{A}}(\mu^3_{\mathcal{A}}(a_3, a_2, a_1)) \pm \mu^3_{\mathcal{A}}(\mu^1_{\mathcal{A}}(a_3), a_2, a_1) \\ & \pm \mu^3_{\mathcal{A}}(a_3, \mu^1_{\mathcal{A}}(a_2), a_1) \pm \mu^3_{\mathcal{A}}(a_3, a_2, \mu^1_{\mathcal{A}}(a_1)); \end{aligned}$$

as $\mu^1_{\mathcal{A}}(a_i) = 0$ we obtain

$$\mu^2_{\mathcal{A}}(\mu^2_{\mathcal{A}}(a_3, a_2), a_2) \pm \mu^2_{\mathcal{A}}(a_3, \mu^2_{\mathcal{A}}(a_2, a_1)) = \pm \mu^1_{\mathcal{A}}(\mu^3_{\mathcal{A}}(a_3, a_2, a_1)).$$

The right-hand side is in the image of $\mu^1_{\mathcal{A}}$, so by passing to cohomology we obtain an equality

$$\left[\mu^2_{\mathcal{A}}(a_3, \mu^2_{\mathcal{A}}(a_2, a_1))\right] = \pm \left[\mu^2_{\mathcal{A}}(\mu^2_{\mathcal{A}}(a_3, a_2), a_1)\right].$$

One can check that the signs actually agree implying that composition of morphisms is associative. □

Definition/Proposition 2.8 ([34, I §1])**.** Let $\mathcal{F}$ be an A_∞-functor between cohomologically unital A_∞-categories $\mathcal{A}$ and $\mathcal{B}$, then there are induced (honest) functors $H^*(\mathcal{F})\colon H^*(\mathcal{A}) \to H^*(\mathcal{B})$ and $H^0(\mathcal{F})\colon H^0(\mathcal{A}) \to H^0(\mathcal{B})$ by sending $[a]$ to $[\mathcal{F}^1(a)]$. If $H^*(\mathcal{F})$ is fully faithful we say that $\mathcal{F}$ is *cohomologically fully faithful.* If $H^*(\mathcal{F})$ is an embedding/equivalence of categories, we say that $\mathcal{F}$ is a *quasi-embedding/equivalence.*

A very useful tool (in homological algebra in general but also particularly in the context of Fukaya categories) is the mapping cone. Unfortunately, mapping cones do not necessarily exist in an arbitrary A_∞-category $\mathcal{A}$. Therefore we have to enlarge $\mathcal{A}$ a little bit so that mapping cones do exist. One way to do this is via twisting.

Definition 2.9 ([8, §3.2])**.** Let $\mathcal{A}$ be an A_∞-category. A *twisted complex* (X, δ^X) in $\mathcal{A}$ comprises the following data: an object X which is a formal sum

$$X = \bigoplus_{i \in I} X_i[k_i]$$

where I is finite, $X_i \in \mathrm{Ob}(\mathcal{A})$ and $k_i \in \mathbb{Z}$; and a differential $\delta^X = (\delta^X_{ij})$ where $\delta^X_{ij} \in \hom^{k_j - k_i + 1}_{\mathcal{A}}(X_i, X_j)$, $i, j \in I$ with $i < j$ (i.e. δ^X is strictly lower triangular), and satisfying the *Maurer–Cartan equation*

$$\sum_{d \geq 1} \mu^d_{\mathcal{A}}(\delta^X, \ldots, \delta^X) = 0$$

i.e. in a more precise way

$$\sum_{\substack{d \geq 1 \\ i = i_0 < i_1 < \cdots < i_d = j}} \mu^d(\delta^X_{i_{k-1} i_k}, \ldots, \delta^X_{i_0 i_1}) = 0$$

holds for all $i, j \in I, i < j$.

Definition 2.10. Let $\mathcal{A}$ be an A_∞-category. The *twisted category* $\mathrm{Tw}\mathcal{A}$ of $\mathcal{A}$ is defined as follows: the objects are twisted complexes in $\mathcal{A}$, i.e. $\mathrm{Ob}(\mathrm{Tw}\mathcal{A}) = \{(X, \delta^X) \mid X \in \mathrm{Ob}(\mathcal{A})\}$. A morphism $a \in \hom^d_{\mathrm{Tw}\mathcal{A}}(X, X')$ of degree d between two objects $X = \bigoplus X_i[k_i]$ and $X' = \bigoplus X'_j\left[k'_j\right]$ is given by $a = (a_{ij})$ where $a_{ij} \in \hom_{\mathcal{A}}^{d+k'_j-k_i}(X_i, X'_j)$. Moreover there are the following composition maps: if $(X_0, \delta^0), \ldots, (X_d, \delta^d) \in \mathrm{Ob}(\mathrm{Tw}\mathcal{A})$, $d \geq 1$ then we have

$$\begin{aligned}\mu^d_{\mathrm{Tw}\mathcal{A}}\colon &\hom_{\mathrm{Tw}\mathcal{A}}((X_{d-1}, \delta^{d-1}), (X_d, \delta^d)) \otimes \ldots \otimes \hom_{\mathrm{Tw}\mathcal{A}}((X_0, \delta^0), (X_1, \delta^1)) \\ &\to \hom_{\mathrm{Tw}\mathcal{A}}((X_0, \delta^0), (X_d, \delta^d))[2-d]\end{aligned}$$

defined via

$$\mu^d_{\mathrm{Tw}\mathcal{A}}(a_d, \ldots, a_1) := \sum_{j_0, \cdots, j_d \geq 0} \mu_{\mathcal{A}}^{d+j_0+\cdots+j_d}((\delta^d)^{\otimes j_d}, a_d, \ldots, (\delta^1)^{\otimes j_1}, a_1, (\delta^0)^{\otimes j_0}).$$

Remark 2.11. Note that, as the δ^i are all strictly lower triangular, the last sum defining $\mu^d_{\mathrm{Tw}\mathcal{A}}$ is finite, similar to the Maurer–Cartan equation. Also by the Maurer–Cartan equation, one can check that $\mu^d_{\mathrm{Tw}\mathcal{A}}$ satisfy the A_∞-relations, so $\mathrm{Tw}\mathcal{A}$ is an A_∞-category.

The twisted category has the important property that one can construct mapping cones.

Definition 2.12. Let $(X, \delta), (X', \delta') \in \mathrm{Ob}(\mathrm{Tw}\mathcal{A})$ and let $a \in \hom_{H^0(\mathrm{Tw}\mathcal{A})}((X, \delta), (X', \delta'))$. Then the *mapping cone* of a in $\mathrm{Tw}\mathcal{A}$ is defined as the twisted complex

$$C(a) = \left(X[1] \oplus X', \begin{pmatrix} \delta & 0 \\ a & \delta' \end{pmatrix}\right).$$

The mapping cone comes with two natural maps, the inclusion of $\iota\colon X' \hookrightarrow C(a)$ and the projection $\pi\colon C(a) \twoheadrightarrow X[1]$; together with a they form an *exact* triangle in $\mathrm{Tw}\mathcal{A}$.

Definition 2.13. Let $\mathcal{A}$ be an A_∞-category, let X, Y and Z be a triple of objects in $\mathcal{A}$ and let $f \in \hom_{H^0(X,Y)}, g \in \hom_{H^0(Y,Z)}$ and $h \in \hom_{H^1(Z,X)}$ be morphisms. If there exists a quasi-isomorphism to the mapping cone $\phi\colon Z \to C(f)$ making the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "="'] & Y \arrow[r, "g"] \arrow[d, "="'] & Z \arrow[r, "h"] \arrow[d, "\phi"] & X[1] \arrow[d, "="] \\ X \arrow[r, "f"] & Y \arrow[r, "\iota"] & C(f) \arrow[r, "\pi"] & X[1] \end{tikzcd}$$

commute, the triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

is said to be an *exact triangle.*

One can show that $\mathcal{A}$ embeds fully faithfully into $\mathrm{Tw}\mathcal{A}$ and that $\mathcal{A}$ *generates* $\mathrm{Tw}\mathcal{A}$ in the sense that repeatedly applying mapping cones and shifts of objects and maps in $\mathcal{A}$, regarded as a subcategory of $\mathrm{Tw}\mathcal{A}$, gives all of $\mathrm{Tw}\mathcal{A}$ [34, I Lem. 3.32]. If $\mathcal{A}$ is cohomologically unital, so is $\mathrm{Tw}\mathcal{A}$ [34, I Lem. 3.24 (i)]. Furthermore, $\mathrm{Tw}\mathcal{A}$ can be shown to be *triangulated* meaning that any morphism in $H^0(\mathcal{A})$ can be completed to an exact triangle and the shift functor $[1]\colon \mathcal{A} \to \mathcal{A}$ is a quasi-equivalence [34, I Lem. 3.28]. The following definition is then sensible.

Definition 2.14. An A_∞-category $\mathcal{A}$ is *triangulated* if the embedding into $\mathrm{Tw}\mathcal{A}$ is a quasi-equivalence.

The twisted category is in some sense "minimal" among those triangulated A_∞-categories containing $\mathcal{A}$, i.e. it is some instance of a triangulated envelope for $\mathcal{A}$ [34, I Lem. 3.34].

Definition 2.15. Let $\mathcal{A}$ be an A_∞-category. We define the *derived category* of $\mathcal{A}$ to be $H^0(\mathrm{Tw}\mathcal{A})$ and denote it as $D\mathcal{A}$.

The derived category $D\mathcal{A}$ is an honest triangulated category and we will later prove an equivalence of triangulated categories between $D\mathcal{W}$ and $D^b\mathrm{Coh}$. For this we will need some theory about module categories as we want to apply the categorical version of the Yoneda Lemma.

Definition 2.16 ([34, I §1] & [11, Def. 1.22])**.** Let $\mathcal{C}$ be a $\mathbb{C}$-linear category, then the *(right) module category* $\mathrm{MOD}(\mathcal{C})$ has as objects $\mathbb{C}$-linear contravariant functors $\rho\colon \mathcal{C}^{op} \to \mathrm{ChVect}(\mathbb{C})$, where $\mathrm{ChVect}(\mathbb{C})$ is the category of chain complexes of $\mathbb{C}$-vector spaces, and as morphisms natural transformations between them.

Example 2.17. We can consider an algebra A as a category with just one object, then the above definition gives the usual category of A-modules.

One can check that if $\mathcal{A}$ is an A_∞-category, $\mathrm{MOD}(\mathcal{A})$ is again an A_∞-category, confer [34, I §1][2]. Moreover, this category is triangulated [34, I §3]. In particular, if A is an A_∞-algebra, the category $\mathrm{MOD}(A)$ is a triangulated A_∞-category. We will use the following version of the Yoneda embedding without proof.

Lemma 2.18 ([8, §3.4.2], Yoneda embedding)**.** *Let $\mathcal{A}$ be an A_∞-category generated by $G_1, \dots, G_r$ and let*

$$\mathcal{G} \coloneqq \bigoplus_{i,j=1}^{r} \hom(G_i, G_j).$$

Then $\mathcal{G}$ is an A_∞-algebra. Moreover, for any $L \in \mathrm{Ob}(\mathcal{A})$, define

$$\mathcal{Y}(L) \coloneqq \bigoplus_{i=1}^{r} \hom(G_i, L),$$

an A_∞-module over $\mathcal{G}$, and for any morphism $a \in \hom(L, L')$, let

$$\mathcal{Y}(a) \in \hom_{\mathrm{MOD}(\mathcal{G})}(\mathcal{Y}(L), \mathcal{Y}(L'))$$

be the morphism induced by composition with a. Then $\mathcal{Y}$ defines an A_∞-functor

$$\mathcal{Y}\colon \mathcal{A} \to \mathrm{MOD}(\mathcal{G})$$

which is a fully faithful quasi-embedding.

[2]In this case one has to replace the functor appearing in Definition 2.16 by an A_∞-functor.

3. Floer Cohomology

Floer cohomology was first introduced by Floer in 1988 to prove Arnold's conjecture about the number of fixed points of a Hamiltonian diffeomorphism on a Lagrangian submanifold. To a pair of Lagrangian submanifolds L_0, L_1, Floer cohomology associates a group $HF(L_0, L_1)$. The Fukaya category, one side of homological mirror symmetry, is the categorification of Floer cohomology, in the sense that its objects are Lagrangian submanifolds and the morphism spaces are given by Floer cohomology groups.

In this section we give a short outline of the construction of Floer cohomology. There are lots of technical details that are omitted for the sake of brevity. The main references are [8, §1 & §2] and [32].

Notation 3.1. During the whole chapter we will assume the symplectic manifold (M, ω) to be compact. Later in the context of wrapped Fukaya categories we will lose this assumption.

3.1. Floer complex, pseudo-holomorphic strips and the Floer differential.

Let L_0, L_1 be two Lagrangian submanifolds of M. For the moment, assume that they intersect transversely (we will deal with the case of non-transverse intersections in §3.4 by slightly perturbing the Lagrangians).

Definition 3.2. Let $\chi(L_0, L_1)$ denote the set of intersection points of L_0 and L_1. Then the *Floer complex* is defined by

$$CF(L_0, L_1) := \bigoplus_{p \in \chi(L_0, L_1)} \mathbb{C} \cdot p.$$

Remark 3.3. This is not the general definition of the Floer complex; in general, one has to use Novikov-coefficients instead of just complex numbers to make the definition of the Floer differential well-defined. This has to do with the energy condition (cf. Def. 3.5). In our case we will exclusively work with *exact* Lagrangians in an exact symplectic manifold; in this case we can work with complex coefficients.

It is a highly non-trivial task to construct a $\mathbb{Z}$-grading on $CF(L_0, L_1)$ and we will devote §3.2 to it.

Now what is the differential $\partial \colon CF(L_0, L_1) \to CF(L_0, L_1)$ of the Floer complex? The idea is that ∂ should count the number of pseudo-holomorphic strips in M bounded by L_0 and L_1. We clarify this below.

Let $J \in \Gamma(\mathrm{End}(TM))$ be an almost complex structure on M which is ω-compatible, i.e. $g_J(-,-) := \omega(-, J(-))$ is a Riemannian metric.

Definition 3.4. Let S be a Riemann surface with complex structure j. A *pseudo-holomorphic curve* in M is (the image of) a map $u \colon S \to M$ such that its differential is complex-linear, i.e.

$$du \circ j = J \circ du. \tag{3}$$

With the usual notion of $\bar{\partial}_J := \frac{1}{2}(d + J \circ d \circ j)$ and using $J^2 = -\mathrm{id}$, we see that (3) is equivalent to the Cauchy–Riemann equation $\bar{\partial}_J u = 0$. Let $z = s + it$ be a complex local coordinate for S, then we can rewrite this as

$$\begin{pmatrix} \frac{\partial u_1}{\partial s} & \frac{\partial u_1}{\partial t} \\ \vdots & \vdots \\ \frac{\partial u_{2n}}{\partial s} & \frac{\partial u_{2n}}{\partial t} \end{pmatrix} + J_u \begin{pmatrix} \frac{\partial u_1}{\partial t} & -\frac{\partial u_1}{\partial s} \\ \vdots & \vdots \\ \frac{\partial u_{2n}}{\partial t} & \frac{\partial u_{2n}}{\partial s} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

which is equivalent to

$$\frac{\partial u}{\partial s} + J_u \frac{\partial u}{\partial t} = 0; \tag{4}$$

in the forthcoming we will mostly use this last form of the Cauchy–Riemann equation.

Let $D := D^2 \setminus \{\pm 1\}$ be the closed unit disk with the points ± 1 removed. It is conformally equivalent to the strip $\mathbb{R}_s \times [0,1]_t$ with coordinates s and t.

Definition 3.5. Let L_0, L_1 be two Lagrangian submanifolds of M with intersection points $p, q \in \chi(L_0, L_1)$. Then a *pseudo-holomorphic strip* from p to q with boundary in L_0 and L_1 is a map $u\colon \mathbb{R} \times [0,1] \to M$ satisfying the Cauchy–Riemann equation (4), the boundary conditions

$$\begin{cases} u(s,0) \in L_0 \text{ and } u(s,1) \in L_1 \text{ for all } s \in \mathbb{R}, \\ \lim_{s \to +\infty} u(s,t) = p \text{ and } \lim_{s \to -\infty} u(s,t) = q; \end{cases} \tag{5}$$

(i.e. u extends to a pseudo-holomorphic curve $u\colon D^2 \to M$ with $u(+1) = p$ and $u(-1) = q$) and the energy condition

$$E(u) := \int_D u^*\omega < \infty. \tag{6}$$

Remark 3.6. The energy condition is one of the assumptions to make the Gromov compactness theorem work (cf. §3.3). Again, the situation simplifies as we will be dealing only with exact Lagrangians in an exact symplectic manifold. Namely, assume $\omega = d\theta$ and $\theta|_{L_i} = df_i$ for $i = 0,1$ and smooth functions f_i. Then we get by Stokes' theorem

$$E(u) = \int_D u^*(d\theta) = \int_{L_0 \cup L_1} u^*\theta = \int_{L_0} df_0 + \int_{L_1} df_1 = f_0(p) - f_0(q) + f_1(q) - f_1(p)$$

which is automatically finite, so we don't have to care too much about the energy condition. Note the different orientations on L_0, L_1 induced by the strip as seen below:

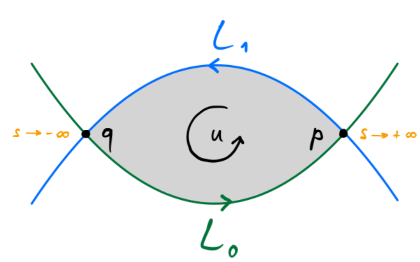


FIGURE 3.1. A pseudo-holomorphic strip u connecting p to q with boundary in L_0 and L_1 with induced orientations

To count the number of pseudo-holomorphic strips bounded by L_0, L_1 we consider the following moduli space: let $[u] \in \pi_2(M, L_0 \cup L_1)$ denote the homotopy class of u. By $\widehat{\mathcal{M}}(p, q, [u], J)$ we denote the moduli space of pseudo-holomorphic strips between $p, q \in \chi(L_0, L_1)$ representing the class $[u]$. The automorphism group of such a strip is $\mathbb{R}$ which acts by rescaling $x.u(s,t) = u(s-x,t)$. Let $\mathcal{M}(p, q, [u], J)$ be the quotient of $\widehat{\mathcal{M}}(p, q, [u], J)$ by this action. As we will see in §3.3 this space is (under some further technical assumptions) a smooth, compact, orientable manifold of dimension $\operatorname{ind}([u]) - 1$, where $\operatorname{ind}([u])$ is the *Maslov index* of the strip u as defined in Def 3.16. Therefore, if we restrict our attention to classes $[u]$ with $\operatorname{ind}([u]) = 1$, $\mathcal{M}(p, q, [u], J)$ is a finite collection of oriented points and the following definition makes sense:

Definition 3.7. The *Floer differential* $\partial\colon CF(L_0, L_1) \to CF(L_0, L_1)$ is the $\mathbb{C}$-linear map defined by

$$\partial(p) := \sum_{\substack{q \in \chi(L_0, L_1) \\ [u] : \operatorname{ind}([u]) = 1}} (\#\mathcal{M}(p, q, [u], J))q.$$

Of course, there are many questions left around this definition, e.g. why $\mathcal{M}(p, q, [u], J)$ is a smooth, compact, orientable manifold of the correct dimension, what happens to non-transverse intersections and why is the differential above even a differential in the sense that $\partial^2 = 0$? We will address some of these problems in the following sections.

For the future discussion we will also have to deal with pseudo-holomorphic maps with more than two boundary components. To be more precise, let $(z_0, \dots, z_k)$ be a tuple of $k+1$ distinct points on the boundary of the disk D^2 in counterclockwise order and let $D = D^2 \setminus \{z_0, \dots, z_k\}$ be the disk with the $k+1$ boundary points removed. Let $C_0, \dots, C_k$ denote the boundary components between the points z_0, z_1 up to z_k, z_0. Note that a neighbourhood around each z_i in D is conformally equivalent to a strip $\mathbb{R}_{s_i} \times [0,1]_{t_i}$.

Definition 3.8. Let $L_0, \dots, L_k$ be Lagrangian submanifolds in M with intersection points $p_i \in \chi(L_{i-1}, L_i)$ for $i = 1, \dots, k$ and $q \in \chi(L_0, L_k)$. A *pseudo-holomorphic disk* with $k+1$ marked points and boundary in $L_0, \dots, L_k$ is a map $u\colon D \to M$ satisfying the Cauchy–Riemann equation (4), the energy condition (6) and the boundary conditions

$$\begin{cases} u|_{C_i} \in L_i \text{ for } i = 0, \dots, k; \\ \lim_{s_i \to +\infty} u(s_i, t_i) = p_{i+1} \text{ and } \lim_{s_i \to -\infty} u(s_i, t_i) = p_i \text{ for } i = 1, \dots, k-1; \\ \lim_{s_0 \to -\infty} u(s_0, t_0) = q \text{ and } \lim_{s_k \to +\infty} u(s_k, t_k) = q. \end{cases}$$

For a given almost-complex structure J on M and a curve class $[u] \in \pi_2(M, L_0 \cup \ldots \cup L_k)$, we denote the moduli space of pseudo-holomorphic disks represented by $[u]$ with $k+1$ marked points and boundary in $L_0, \dots, L_k$ by $\widehat{\mathcal{M}}(p_1, \dots, p_k, q, [u], J)$. The moduli space of pseudo-holomorphic disks modulo automorphisms is denoted by $\mathcal{M}(p_1, \dots, p_k, q, [u], J)$.

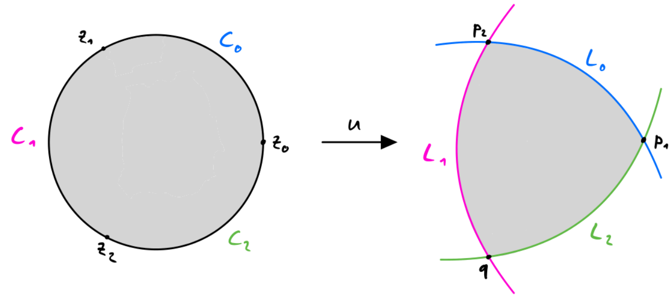


FIGURE 3.2. A pseudo-holomorphic disk with three marked points

3.2. **Index of a strip, graded Lagrangians and a grading for the Floer complex.** In this section we will first define the index of a pseudo-holomorphic strip which is used in the definition of the Floer differential. Secondly we will define a $\mathbb{Z}$-grading on the Floer complex for which we need to equip the Lagrangian submanifolds with some extra structure.

Notation 3.9. In this section we will always assume that the intersection of two Lagrangian submanifolds is transverse.

Definition 3.10 ([7, §1.2])**.** Let (V, Ω) be a symplectic vector space. The manifold of all Lagrangian subspaces is called the *Lagrangian Grassmannian* and is denoted $\mathrm{LGr}(V)$. If $V = \mathbb{R}^{2n}$ we may just write $\mathrm{LGr}(n)$.

Lemma 3.11 ([7, §1.2])**.** *The Lagrangian Grassmannian* $\mathrm{LGr}(n)$ *can be identified with* $U(n)/O(n)$ *showing that it is a smooth manifold of dimension* $\frac{1}{2}n(n+1)$.

Proof. We show that $U(n)$ acts on $\mathrm{LGr}(n)$ (considered as a set) transitively with stabiliser group $O(n)$. Note that on $(\mathbb{R}^{2n}, \Omega)$ we have a metric $g(-,-) = \langle -,- \rangle$ and an almost complex structure $J\colon (\boldsymbol{x}, \boldsymbol{y}) \mapsto (-\boldsymbol{y}, \boldsymbol{x})$ compatible with Ω, ie $\Omega(-,-) = g(J(-),-)$. An automorphism of $\mathbb{R}^{2n}$ preserving two of these structures will automatically preserve the third structure, implying that such an automorphism is an element of $U(n) = Sp(2n) \cap O(n)$. Therefore, $U(n)$ acts on $\mathrm{LGr}(n)$ and if the Lagrangian plane λ is mapped to λ' with orthogonal coordinates $(\boldsymbol{x}, \boldsymbol{y}), (\boldsymbol{x}', \boldsymbol{y}')$ respectively, then $J(\boldsymbol{x}, \boldsymbol{y})$ is mapped to $J(\boldsymbol{x}', \boldsymbol{y}')$. As $J\lambda$ is orthogonal to λ, we see that $O(n)$ is the stabiliser group. Noting that $\dim(U(n)) = n^2$ and $\dim(O(n)) = \frac{1}{2}n(n-1)$, we get $\dim(\mathrm{LGr}(n)) = \frac{1}{2}n(n+1)$. □

Example 3.12. Consider $\mathbb{R}^2$ with the standard symplectic form. Then every 1-dimensional subspace is Lagrangian. Therefore, $\mathrm{LGr}(1) \cong \mathbb{R}P^1$.

The following map will be very important in the ongoing construction.

Definition/Proposition 3.13 ([7, §1.3 & §1.4])**.** The map

$$\det{}^2\colon \mathrm{LGr}(n) = U(n)/O(n) \to S^1 \tag{7}$$

induces an isomorphism on fundamental groups. The *Maslov index* of a loop γ in $\mathrm{LGr}(n)$ is the winding number of its image under this map and is denoted by $\mu(\gamma)$.

Proof. Let $S\mathrm{LGr}(n)$ be the set of Lagrangian submanifolds λ of $(\mathbb{R}^{2n}, \Omega)$ with $\det^2 \lambda = 1$. We have $S\mathrm{LGr}(n) = SU(n)/SO(n)$ and we get the following commutative diagram:

$$\begin{tikzcd}
SO(n) \arrow[r] \arrow[d] & O(n) \arrow[r, "\det"] \arrow[d] & S^0 \arrow[d] \\
SU(n) \arrow[r] \arrow[d] & U(n) \arrow[r, "\det"] \arrow[d] & S^1 \arrow[d, "z\mapsto z^2"] \\
S\mathrm{LGr}(n) \arrow[r] & \mathrm{LGr}(n) \arrow[r, "\det^2"] & S^1
\end{tikzcd}$$

The left column induces a long exact homotopy sequence

$$\ldots \to \pi_1(SU(n)) \to \pi_1(S\mathrm{LGr}(n)) \to \pi_0(SO(n)) \hookrightarrow \pi_0(SU(n)) \to \cdots$$

As $SU(n)$ is simply connected, we get $\pi_1(SU(n)) = 0$. Taking homotopy of the bottom row then yields

$$0 \to \pi_1(\mathrm{LGr}(n)) \xrightarrow{\det^2_*} \pi_1(S^1) \to \pi_0(S\mathrm{LGr}(n)) \hookrightarrow \pi_0(\mathrm{LGr}(n)) \to \cdots,$$

so $\det^2_*$ is an isomorphism on fundamental groups. □

Lemma 3.14. *$Sp(2n,\mathbb{R})$ acts 2-transitively on transverse Lagrangian subspaces, ie for $(\lambda_0, \lambda_1), (\lambda_0', \lambda_1')$ two pairs of transverse Lagrangian subspaces, there exists $S \in Sp(2n,\mathbb{R})$ such that $(\lambda_0', \lambda_1') = (S\lambda_0, S\lambda_1)$.*

Proof. Because of the transversality we can find bases $(e_1, \ldots, e_n)$ for λ_0, $(e_1', \ldots, e_n')$ for λ_0', and $(f_1, \ldots, f_n)$ for λ_1, $(f_1', \ldots, f_n')$ for λ_1' such that $(e_1, \ldots, e_n, f_1, \ldots, f_n)$ and $(e_1', \ldots, e_n', f_1', \ldots, f_n')$ are symplectic bases for $(\mathbb{R}^{2n}, \Omega)$. Then the map S defined via $e_i \mapsto e_i'$, $f_i \mapsto f_i'$ for $i = 1, \ldots, n$, is an element of $Sp(2n,\mathbb{R})$ and satisfies $(\lambda_0', \lambda_1') = (S\lambda_0, S\lambda_1)$. □

Definition 3.15 ([8, §1.3])**.** Let $\lambda_0, \lambda_1 \in \mathrm{LGr}(n)$ be transverse Lagrangians and identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$. By Lemma 3.14 above, there exists $S \in Sp(2n,\mathbb{R})$ mapping (λ_0, λ_1) to $(\mathbb{R}^n, (i\mathbb{R})^n) \subseteq (\mathbb{C}^n)^2$. The subspaces $\lambda_t := S^{-1}((e^{-\pi i t/2}\mathbb{R})^n)$, for $t \in [0,1]$, are again Lagrangian, so λ_t defines a path in $\mathrm{LGr}(n)$. Its homotopy class is well-defined (ie it doesn't depend on the choice of S) and we call it *canonical short path* from λ_0 to λ_1, denoted $\gamma_{\lambda_0,\lambda_1}$.

Be aware that $\gamma_{\lambda_0,\lambda_1}$ is *clockwise* orientation in $\mathrm{LGr}(n)$—the orientation of the path plays an important role as we will see in Example 3.19. We are now sufficiently prepared to define the index of a pseudo-holomorphic strip:

Definition 3.16 ([8, §1.3])**.** Let L_0, L_1 be two transverse Lagrangian submanifolds of M, $p, q \in L_0 \cap L_1$. For brevity, let $\gamma_p := \gamma_{T_pL_0,T_pL_1}$ and $\gamma_q := \gamma_{T_qL_0,T_qL_1}$ be canonical short paths. Let $u\colon \mathbb{R}_s \times [0,1] \to M$ be a pseudo-holomorphic strip connecting p to q and let $l_i = (u|_{\mathbb{R}_s\times\{i\}})^*TL_i$, for $i = 0, 1$, be the path from T_pL_i to T_qL_i, oriented with s coordinate going from $+\infty$ to $-\infty$. By fixing a trivialisation of u^*TM, we can identify γ_p, γ_q and l_i as paths in $\mathrm{LGr}(n)$. Then the *index* of u is defined as the Maslov index

$$\mathrm{ind}(u) = \mu(-\gamma_q \circ l_1 \circ \gamma_p \circ -l_0)$$

where we use the usual notation $-\eta(t) := \eta(1-t)$ for any loop η.

Remark 3.17. In fact, $\mathrm{ind}(u)$ only depends on the homotopy class $[u] \in \pi_2(M, L_0 \cup L_1)$.

Remark 3.18. The index commutes with addition in π_2, i.e. for $[u_1], [u_2] \in \pi_2(M, L_0, L_1)$ we have $\mathrm{ind}([u_1] + [u_2]) = \mathrm{ind}([u_1]) + \mathrm{ind}([u_2])$.

Example 3.19. Let us consider the case $M = \mathbb{R}^2$ and u as depicted in Figure 3.6 above. By fixing a trivialisation we get the following identifications in $\mathbb{R}^2$:

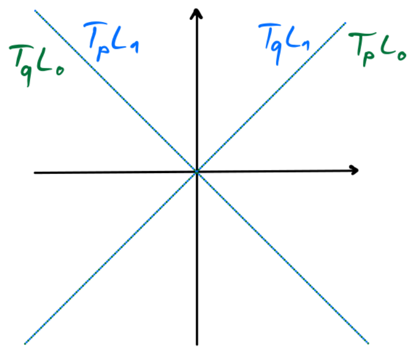


FIGURE 3.3. The tangent spaces of intersecting Lagrangians inside $T_pM \hat{=} T_qM$

Here, $\mathrm{LGr}(1)$ is naturally identified with $\mathbb{R}P^1$. Then the index of u is the Maslov index of the following paths which we see corresponds to a single rotation, so we conclude $\mathrm{ind}(u) = 1$.

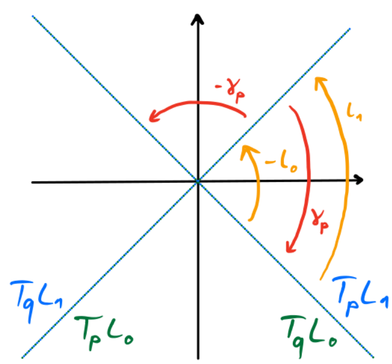


FIGURE 3.4. The paths involved in the definition of $\mathrm{ind}(u)$

Remark 3.20. Note that if L_0 and L_1 are transverse and $u: \mathbb{R} \times [0,1] \to M$ is nonconstant then $\mathrm{ind}(u) \geq 1$.

Remark 3.21. Analogously to the definition above, one can also define the Maslov index of a pseudo-holomorphic disk with more marked points and boundary components by concatenation of the corresponding paths obtained from the boundary components and the canonical short paths.

We will now establish a $\mathbb{Z}$-grading on the Floer complex. This does not work in full generality but we need some extra conditions and structure. The resources used are [8, §1.3] and [32, Lecture 23].

One can "globalise" the Lagrangian Grassmannian to the Lagrangian Grassmannian bundle $\mathrm{LGr}(TM)$, a subbundle of the Grassmannian bundle $\mathrm{Gr}(TM)$ which in each fibre restricts to $\mathrm{LGr}(T_pM)$. The general idea is to make the $\det^2$ function (7) into a globally defined function on $\mathrm{LGr}(TM)$. Recall that the determinant is the unique multilinear map which is alternating and normed (in the sense that $\det(I) = 1$). Therefore we want to have a section δ of $(\Lambda^n_{\mathbb{C}}T^*M)^{\otimes 2}$ (we need the tensor power because we want the square of the determinant) and define a function

$$\Delta = \frac{\delta}{|\delta|}\colon \mathrm{LGr}(TM) \to S^1.$$

For this to be well-defined, we need δ to be nowhere vanishing, ie δ trivialises the line bundle $(\Lambda^n_{\mathbb{C}}T^*M)^{\otimes 2}$ which is equivalent to $c_1((\Lambda^n_{\mathbb{C}}T^*M)^{\otimes 2}) = -2c_1(TM) = 0$. Hence we have found a necessary condition we need, namely

$$2c_1(TM) = 0.$$

Under this assumption, the function Δ is well-defined and indeed restricts to $\det^2$ on every fibre. Note that the condition is for example satisfied by every Calabi–Yau manifold.

Let L be a Lagrangian submanifold of M. Then there exists a unique lift

$$\begin{tikzcd} & \mathrm{LGr}(TM) \arrow[d] \\ L \arrow[ur, "\widetilde{i}"] \arrow[r, hook, "i"] & M \end{tikzcd}$$

and we define the function Δ_L obtained by the composition

$$\Delta_L\colon L \xrightarrow{\widetilde{i}} \mathrm{LGr}(TM) \xrightarrow{\Delta} S^1, \quad p \mapsto \Delta(T_pL).$$

Definition 3.22. The map Δ_L induces a map $(\Delta_L)_*\colon \pi_1(L) \to \pi_1(S^1) \cong \mathbb{Z}$. Under the Hurewicz map and by the Universal Coefficient Theorem, the map $(\Delta_L)_*$ is a class in $H^1(L;\mathbb{Z})$. It is called the *Maslov class* of L, denoted μ_L.

Let $\widetilde{\mathrm{LGr}}(TM)$ be the fibrewise universal cover of $\mathrm{LGr}(TM)$. As we will now see, the Maslov class is the obstruction to lifting the section $p \mapsto T_pL$ of $\mathrm{LGr}(TM)$ to $\widetilde{\mathrm{LGr}}(TM)$.

Consider the following diagram

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p"] \\ \Delta_L\colon L \arrow[ur, dashed, "\tilde{\Delta}_L"] \arrow[r] & S^1 \end{tikzcd}$$

where p is the universal cover for S^1. By standard covering theory, the lift $\tilde{\Delta}_L$ exists if and only if $(\Delta_L)_*(\pi_1(L)) \subseteq p_*(\pi_1(\mathbb{R})) = 0$ which is equivalent to $\mu_L = 0$. Note that if a lift exists, the set of all possible lifts forms a torsor over $\mathbb{Z}$ (simply by addition). We obtain the following

Definition/Proposition 3.23. Assume $2c_1(TM) = 0$ and fix a map $\Delta\colon \mathrm{LGr}(TM) \to S^1$ as above. Let L be a Lagrangian submanifold of M with $\mu_L = 0$. Then there exists a map $\tilde{\Delta}_L\colon L \to \mathbb{R}$ such that $\exp(2\pi i \tilde{\Delta}_L) = \Delta_L$. The pair $(L, \tilde{\Delta}_L)$ is called a *graded Lagrangian submanifold* of M. We also sometimes denote it by $\tilde{L} \in \widetilde{\mathrm{LGr}}(TM)$.

Example 3.24. In the case where $M = \mathbb{R}^2$, we have $\mathrm{LGr}(TM) \simeq S^1$ and $\widetilde{\mathrm{LGr}}(TM) \simeq \mathbb{R}$. A graded Lagrangian comprises the data of a Lagrangian together with an integer determining the lift.

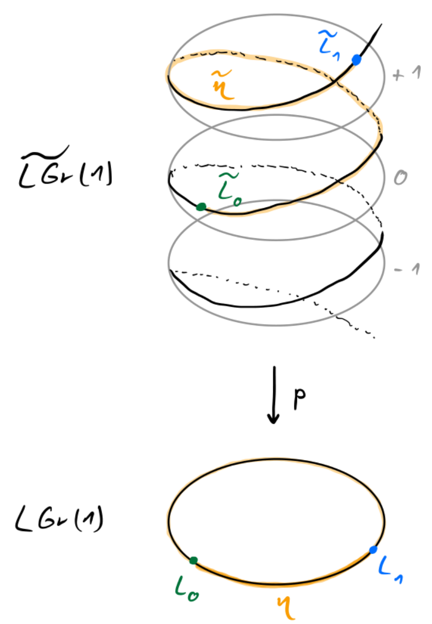


FIGURE 3.5. The universal cover $\widetilde{\mathrm{LGr}}(1) \to \mathrm{LGr}(1)$; there is a preferred path $\tilde{\eta}$ between lifts $\tilde{L}_0$ and $\tilde{L}_1$

Finally we are readily prepared to define a grading on the Floer complex $CF(L_0, L_1)$.

Definition 3.25. Let $(L_0, \tilde{\Delta}_{L_0}), (L_1, \tilde{\Delta}_{L_1})$ be two graded Lagrangian submanifolds and let $p \in L_0 \cap L_1$. As $\widetilde{\mathrm{LGr}}(T_pM)$ is simply connected, there exists a unique homotopy class of paths from $T_p\tilde{L}_0$ to $T_p\tilde{L}_1$, say $\tilde{\eta}_p$. By composition with the universal cover map this becomes a path η_p in $\mathrm{LGr}(T_pM)$ from T_pL_0 to T_pL_1. Recall that γ_p is the canonical short path from T_pL_0 to T_pL_1. Then we define the *degree* of p as

$$\deg(p) := \mu(-\gamma_p \circ \eta_p).$$

Proposition 3.26. *Let u be a pseudo-holomorphic strip connecting p to q with boundary L_0 and L_1. Then*

$$\mathrm{ind}(u) = \deg(q) - \deg(p).$$

Proof. Consider the following diagram of path concatenations:

$$\begin{tikzcd} T_qL_0 \arrow[r, "-l_0"] \arrow[d, bend left, "\eta_q"] & T_pL_0 \arrow[d, bend left, "\gamma_p"] \\ T_qL_1 \arrow[u, bend left, "-\gamma_q"] & T_pL_1 \arrow[l, "l_1"] \arrow[u, bend left, "-\eta_p"] \end{tikzcd}$$

Taking the Maslov index of the big outer circle gives $\mathrm{ind}(u)$, whereas the Maslov indices of the small circles yield $\deg(q)$ for the left and $\deg(p)$ for the right one. As the path $(-\eta_q) \circ l_1 \circ \eta_p \circ (-l_0)$ is nullhomotopic, we get the desired result. Also note that the ambiguity in the choice of the lifts $\tilde{\Delta}_{L_{0,1}}$ has no influence on this calculation as $\tilde{\Delta}_{L_0}(p) - \tilde{\Delta}_{L_1}(p) = \tilde{\Delta}_{L_0}(q) - \tilde{\Delta}_{L_1}(q)$, so they cancel out by taking the difference $\deg(q) - \deg(p)$. □

Corollary 3.27. *The Floer differential as defined in Def 3.7 has degree 1.*

Remark 3.28. If we consider a pseudo-holomorphic disk u with $k+1$ marked points we get analogously

$$\mathrm{ind}(u) = \deg(q) - \sum_{i=1}^{k} \deg(p_i).$$

3.3. **The moduli space of pseudo-holomorphic strips.** The moduli space of pseudo-holomorphic strips $\mathcal{M}(p_1, \ldots, p_k, q, [u], J)$ is a key ingredient in the definition of the Floer differential (Def 3.7) and will play an equally important role in the definition of higher product operations below. In this section we present some technical properties and issues.

Firstly we want to understand how to construct $\mathcal{M}(p_1, \ldots, p_k, q, [u], J)$ not just as a set but as a smooth manifold, following [32, Lectures 15 & 25]. Let

$$\mathcal{B}^\infty = \{\text{smooth maps } u \colon D \to M \text{ satisfying the boundary conditions as in (5)}\}$$

where D is the disk D^2 with $k+1$ marked points; note that this is not a manifold in the usual sense, but a Banach manifold, the infinite-dimensional generalisation of a usual manifold, i.e. a space locally modeled by Banach spaces, see e.g. [26]. Its tangent space at a point u is given by

$$T_u\mathcal{B}^\infty = \left\{ \substack{\text{smooth sections } \xi\in\Gamma(D,u^*TM) \text{ such that the restrictions to the two} \\ \text{boundary components of } D \text{ are elements of } \Gamma(D,u^*TL_0),\ldots,\Gamma(D,u^*TL_k)} \right\}.$$

There is an infinite-rank (Banach) vector bundle $\mathcal{E}^\infty$ over $\mathcal{B}^\infty$ with fibre over $u \in \mathcal{B}^\infty$ given by $\Omega^{0,1}_D(u^*TM)$. The Cauchy–Riemann operator $\bar{\partial}_J$ then defines a section

$$\begin{tikzcd} \mathcal{E}^\infty \arrow[d] \\ \mathcal{B}^\infty \arrow[u, bend right, "\bar{\partial}_J"'] \end{tikzcd}$$

and $\widehat{\mathcal{M}}(p_1, \ldots, p_k, q, [u], J)$ is a component of the vanishing locus of this section. Modding out its automorphism group gives us $\mathcal{M}(p_1, \ldots, p_k, q, [u], J)$. If $\bar{\partial}_J$ intersects the zero section transversely, $\mathcal{M}(p_1, \ldots, p_k, q, [u], J)$ will be a smooth manifold.

Definition 3.29. Let V, W be two Banach spaces. A bounded linear operator $T\colon V \to W$ is called *Fredholm* if $T(V)$ is closed in W and $\ker(T)$ as well as $\operatorname{coker}(T)$ are finite-dimensional. We define the *index* of a Fredholm operator to be

$$\operatorname{ind}(T) := \dim(\ker(T)) - \dim(\operatorname{coker}(T)).$$

The differential of the Cauchy–Riemann operator gives a map

$$d_u\bar{\partial}_J\colon T_u\mathcal{B}^\infty \to T_{(u,0)}\mathcal{E}^\infty = T_u\mathcal{B}^\infty \oplus \mathcal{E}^\infty_u;$$

composing this with the projection $\pi_u\colon T_{(u,0)}\mathcal{E}^\infty \to \mathcal{E}^\infty_u$ gives the *linearised Cauchy–Riemann operator* $D_u := \pi_u \circ d_u\bar{\partial}_J$. It can be shown that D_u is a Fredholm operator (cf. [29, §3]). If D_u is surjective at every point, the dimension of $\widehat{\mathcal{M}}(p_1, \dots, p_k, q, [u], J)$ is given by $\operatorname{ind}(D_u)$. One can always achieve this surjectivity assumption by replacing the almost-complex structure J with a family of ω-compatible almost-complex structures $\{J_t\}_{t\in[0,1]}$ (cf. [29, §3]). Recall that there is a unique automorphism of the disk once you have fixed three points on the boundary. Therefore, if you have $k+1$ marked points they can be varied in a $(k-2)$-dimensional space, so we have

$$\dim \mathcal{M}(p_1, \dots, p_k, q, [u], J) = k - 2 + \operatorname{ind}(D_u).$$

How can we calculate $\operatorname{ind}(D_u)$? The first step is to realise that there exists a homotopy through Fredholm operators between D_u and

$$\bar{\partial}_u\colon \Gamma(E, F) \to \Omega^{0,1}(E)$$

where we write E for u^*TM and F for the Lagrangian subbundle u^*TL_i over the boundary, i.e. (E, F) is a pair of vector bundles over $(D, \partial D)$ (cf. [29, §3.3]).

For now, assume that D is unpunctured. As D is contractible, we have $E \cong \mathbb{C}^n$. In fact, one can prove that (E, F) splits into pairs of line bundles of the form $(\mathbb{C}, z^{\kappa_i/2}\mathbb{R})$ for integers κ_i summing up to the Maslov index $\sum_{i=1}^n \kappa_i = \operatorname{ind}(u)$. In this one-dimensional case one can directly calculate $\operatorname{ind}(\bar{\partial}_u) = \kappa_i + 1$, as is done in [30]. This gives $\operatorname{ind}(\bar{\partial}_u) = \operatorname{ind}(u) + n$. The existence of a $\mathbb{Z}$-grading implies that the loop defined by the boundary of u lifts to $\widetilde{\mathrm{LGr}}$ where it becomes nullhomotopic, thus $\operatorname{ind}(u) = 0$, confer [34, II Lem. 11.12].

What can we do if u has marked boundary points? Suppose u has marked points $p_1, \dots, p_k, q$. The idea is to glue disks v_i and w with a single marked point each to u at p_i respectively q to transform u into an unpunctured disk u'.

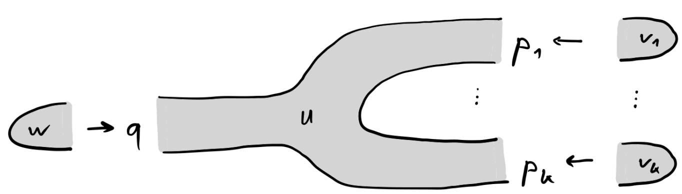


FIGURE 3.6. Construction of the unpunctured disk u' by gluing disks with a single marked point to u

The v_i are defined as follows: let L_i and L_i' be the (graded) Lagrangians intersecting at p_i. As in Definition 3.25, there is a distinguished path η_{p_i} between $T_{p_i}L_i$ and $T_{p_i}L_i'$

inside $\mathrm{LGr}(T_{p_i}M)$ coming from the unique path between the graded Lagrangian lifts in the universal cover. We now want that the boundary of v_i corresponds to η_{p_i}. Similarly, at the point q we get a distinguished path η_q between T_qL_k and T_qL_0 and we want the boundary of the disk w to correspond to $-\eta_q$. We then have $\mathrm{ind}(\bar\partial_{v_i}) = \deg(p_i)$ and $\mathrm{ind}(\bar\partial_w) = n - \deg(q)$. By additivity of the index under gluing (see [34, II (11.9)]) we get

$$\mathrm{ind}(\bar\partial_{u'}) = \mathrm{ind}(\bar\partial_u) + \sum_{i=1}^{k} \mathrm{ind}(\bar\partial_{v_i}) + \mathrm{ind}(\bar\partial_w) = \mathrm{ind}(\bar\partial_u) + \sum_{i=1}^{k} \deg(p_i) + n - \deg(q).$$

Combining this with the result for the unpunctured disk above we obtain

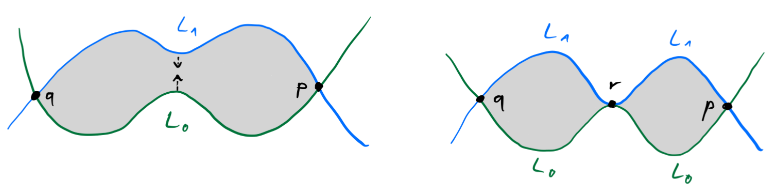
$$\mathrm{ind}(\bar\partial_u) = \deg(q) - \sum_{i=1}^{k} \deg(p_i).$$

In particular, we get as the dimension for the moduli space of pseudo-holomorphic disks

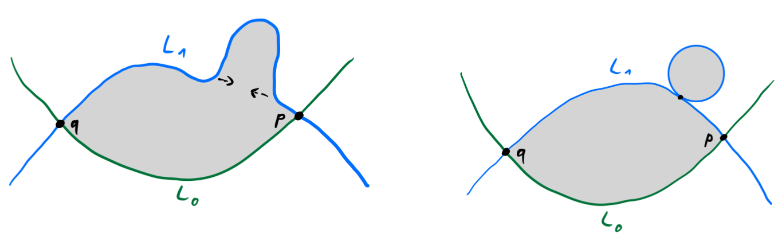
$$\dim \mathcal{M}(p_1, \dots, p_k, q, [u], J) = k - 2 + \deg(q) - \sum_{i=1}^{k} \deg(p_i). \tag{8}$$

The issue of compactness of the moduli spaces is addressed by *Gromov's Compactness Theorem*. It states that a sequence of pseudo-holomorphic curves with uniformly bounded energy admits a subsequence which converges to a nodal tree of pseudo-holomorphic curves (cf. [21, Thm. 1.5.B]). In the case of $\mathcal{M}(p, q, [u], J)$ there are three different types of boundary curves possible:

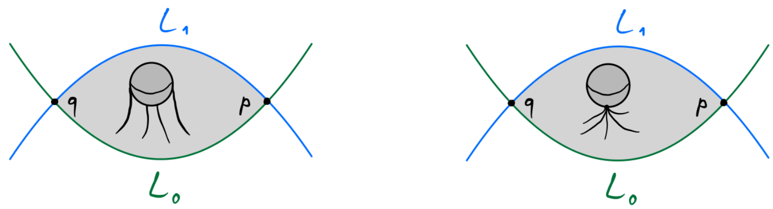
(1) strip breaking



(2) disk bubbling



(3) sphere bubbling



For the proof that the Floer differential squares to zero which we will present in §3.5, it is important that neither disk nor sphere bubbling may occur. To ensure this, we have to impose the extra condition that $[\omega] \cdot \pi_2(M, L_i) = 0$ for $i = 0, 1$ (cf. [8, §1.4]).

Remark 3.30. Note that for an exact Lagrangian submanifold L, the condition $[\omega] \cdot \pi_2(M, L) = 0$ is automatically satisfied by Stokes' Theorem.

The moduli space $\mathcal{M}(p_1, \ldots, p_k, q, [u], J)$ carries a natural orientation if we equip the Lagrangian submanifolds with a *Pin-structure*. The reader is referred to [34, §11] for details. A Lagrangian submanifold equipped with a grading $\tilde{\Delta}_L$ and a Pin-structure is sometimes also called a *Lagrangian brane*.

3.4. **Hamiltonian perturbations.** Now we want to explain briefly how we can remedy the issue of non-transverse intersections. A very important special case of this is the Floer cohomology $HF(L, L)$ of a Lagrangian with itself. The basic idea is to introduce a Hamiltonian perturbation to make the intersection transverse. We are following [8, §1.4] for this discussion.

Let $H \in C^\infty([0,1] \times M, \mathbb{R})$ be a Hamiltonian function. Now instead of considering the usual Cauchy–Riemann equation (4), we introduce a perturbation by the Hamiltonian vector field X_H:

$$\frac{\partial u}{\partial s} + J_u \left(\frac{\partial u}{\partial t} - X_H(t, u) \right) = 0. \tag{9}$$

(In general, there is also a time-dependence of the almost-complex structure to achieve regularity of the linearised Cauchy–Riemann operator, but we omit this here for simplicity.) The generators of the Floer complex $CF(L_0, L_1)$ are now no longer intersection points in $L_0 \cap L_1$ but instead perturbed intersection points in $L_0 \cap (\phi_H^1)^{-1} L_1$ where ϕ_H^1 is the time-1-flow corresponding to X_H.

Change of Notation 3.31. From now on, we denote by $\chi(L_0, L_1)$ the intersection $L_0 \cap (\phi_H^1)^{-1} L_1$ for a Hamiltonian H making the intersection transverse.

We can rewrite the perturbed Cauchy–Riemann equation (9) as follows: define $\tilde{u}(s,t) := (\phi_H^t)^{-1}(u(s,t))$ where ϕ_H^t is the time-t-flow of X_H. We have

$$\frac{\partial \tilde{u}}{\partial t} = (\phi_H^t)_*^{-1} \left(\frac{\partial u}{\partial t} - X_H \right)$$

and, using that ϕ_H^t is a diffeomorphism, equation (9) is equivalent to

$$\frac{\partial \tilde{u}}{\partial s} + \tilde{J}_{\tilde{u}} \frac{\partial \tilde{u}}{\partial t} = 0$$

where $\tilde{J} = (\phi_H^t)_*^{-1} J$; this is again a non-perturbed Cauchy–Riemann equation, so instead of using perturbed solutions of (9), we can instead consider pseudo-holomorphic strips with respect to $\tilde{J}$ and with boundary on L_0 and $(\phi_H^1)^{-1} L_1$.

Remark 3.32. Sometimes it will be convenient to consider $\chi(L_0, L_1)$ as $\phi_H^1(L_0) \cap L_1$ instead of $L_0 \cap (\phi_H^1)^{-1} L_1$. The discussion in §3.5 will show that Floer cohomology is invariant under Hamiltonian isotopies so this change of view has no influence on our results.

Example 3.33. Consider the cylinder $T^*S^1 = \mathbb{R} \times S^1$ with coordinates (r, θ) and standard Liouville form $r\mathrm{d}\theta$, so $\omega = \mathrm{d}r \wedge \mathrm{d}\theta$; and consider a Lagrangian cotangent fibre L. Choose as Hamiltonian function $H = r^2$. Then X_H is the vector field such that $\omega(-, X_H) = 2r\mathrm{d}r$, implying that $X_H = 2r\frac{\partial}{\partial\theta}$. The Hamiltonian flow then calculates as $\phi_H^t(r, \theta) = (r, \theta + 2tr)$. Therefore, $L' := \phi_H^1(L)$ is a "wrapping" around the cylinder:

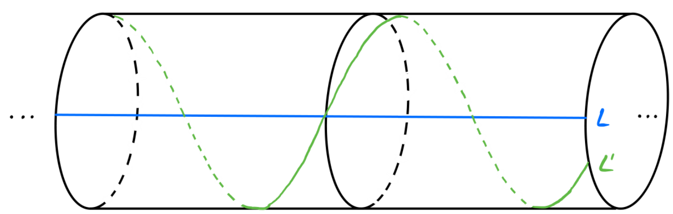


FIGURE 3.7. Wrapping a Lagrangian L around the cylinder

3.5. $\partial^2 = 0$ **and independence of choices.** We will now sketch a proof that the Floer differential squares to zero, following [8, §1.5], via a nice cobordism argument.

Let L_0, L_1 be two Lagrangian submanifolds of M and assume they intersect transversely (otherwise introduce a suitable Hamiltonian perturbation). Moreover, assume that $[\omega] \cdot \pi_2(M, L_i) = 0$ for $i = 0, 1$, so that no disk or sphere bubbling can occur. Let $p, q \in \chi(L_0, L_1)$ and let $[u]$ be a class representing a pseudo-holomorphic strip from p to q with boundary in L_0, L_1. By Gromov's Compactness Theorem, $\mathcal{M}(p, q, [u], J)$ can be compactified to a space $\overline{\mathcal{M}}(p, q, [u], J)$ whose boundary consists of broken strips connecting p and q. A broken strip as in Figure 1 is an element of $\mathcal{M}(p, r, [u_1], J) \times \mathcal{M}(r, q, [u_2], J)$ with $[u_1] + [u_2] = [u]$. Let us assume $\mathrm{ind}([u]) = 2$; then by Remarks 3.18 and 3.20, we must have $\mathrm{ind}([u_1]) = \mathrm{ind}([u_2]) = 1$. On the other hand, it can be shown that any broken strip arises as the limit of a family of pseudo-holomorphic strips with index two (cf. [29, App. A]). Therefore, we find

$$\partial\overline{\mathcal{M}}(p, q, [u], J) = \coprod_{\substack{r \in \chi(L_0, L_1) \\ [u_1] + [u_2] = [u] \\ \mathrm{ind}([u_1]) = \mathrm{ind}([u_2]) = 1}} (\mathcal{M}(p, r, [u_1], J) \times \mathcal{M}(r, q, [u_2], J)).$$

Moreover, it can be shown that the orientations in the spaces above agree up to an overall sign. Note that $\overline{\mathcal{M}}(p, q, [u], J)$ is a compact one-dimensional manifold. Therefore, the signed number of boundary points equals zero and we get

$$\sum_{\substack{r\in\chi(L_0,L_1)\\ [u_1]+[u_2]=[u]\\ \mathrm{ind}([u_1])=\mathrm{ind}([u_2])=1}} (\#\mathcal{M}(p,r,[u_1],J))\cdot(\#\mathcal{M}(r,q,[u_2],J))=0.$$

But the left-hand side of the above equation is just the coefficient of q in $\partial^2(p)$ for a fixed $[u]$. Summing over all possible $[u]$ then gives:

Proposition 3.34. *Let L_0, L_1 be two Lagrangian submanifolds of M intersecting transversely with $[\omega]\cdot\pi_2(M,L_i)=0$ for $i=0,1$. Then the Floer differential $\partial\colon CF(L_0,L_1)\to CF(L_0,L_1)$ satisfies $\partial^2=0$.* □

Therefore, under the above assumptions, it makes sense to define Floer cohomology as

$$HF(L_0,L_1) := H^*(CF(L_0,L_1),\partial).$$

If the Lagrangians are equipped with a grading, we obtain a $\mathbb{Z}$-grading on $HF(L_0,L_1)$.

Remark 3.35. If $[\omega]\cdot\pi_2(M,L_i)\neq 0$ it can in fact happen that $\partial^2\neq 0$, see [8, Ex. 1.11] for an example.

Our definition of Floer cohomology involved several choices, namely the almost-complex structure J on M and the Hamiltonian perturbation H. We would like to show that Floer cohomology is in fact, up to isomorphism, independent of these choices. Again, we are following [8, §1.5] here.

Let $(H,J),(H',J')$ be two choices of Hamiltonian perturbation and almost-complex structure and assume L_0, L_1 intersect transversely with respect to both perturbations.. The key idea is to interpolate between the two choices via a continuation map which counts strip similarly to the pseudo-holomorphic strips before but now with a moving condition. Recall the classical result that the space $\mathcal{J}(M,\omega)$ of ω-compatible almost-complex structures is contractible. Clearly also the space of Hamiltonian functions is contractible. Therefore, we can find a smooth family $(H(\tau),J(\tau))$ for $\tau\in[0,1]$ with $(H(0),J(0))=(H,J)$ and $(H(1),J(1))=(H',J')$.

Definition 3.36. Let $\tau\colon\mathbb{R}\to[0,1]$ be a smooth function with $\tau(s)=1$ for $s\ll 0$ and $\tau(s)=0$ for $s\gg 0$. The *continuation map* between perturbation data (H,J) and (H',J') is the map $F\colon CF(L_0,L_1,H,J)\to CF(L_0,L_1,H',J')$ defined via: if $p\in\chi(L_0,L_1,H)$ and $p'\in\chi(L_0,L_1,H')$, then the coefficient of p' in $F(p)$ is the number of solutions $u\colon\mathbb{R}\times[0,1]\to M$ satisfying

$$\frac{\partial u}{\partial s}+J_u(\tau(s),t)\left(\frac{\partial u}{\partial t}-X_H(\tau(s),t,u)\right)=0, \tag{10}$$

the boundary conditions

$$\lim_{s\to+\infty}u(s,t)=p \text{ and } \lim_{s\to-\infty}u(s,t)=p',$$

the energy condition

$$E(u)+\int_{\mathbb{R}\times[0,1]}\frac{\partial H(\tau(s),s,t)}{\partial s}\,\mathrm{d}s\mathrm{d}t<\infty \tag{11}$$

and $\operatorname{ind}(u) = 0$.

One can check that under the usual no-bubbling-assumptions F is a chain map, again using a cobordism argument, this time considering the moduli space of index 1 solutions to (10). Considering the continuation map $F' \colon CF(L_0, L_1, H', J') \to CF(L_0, L_1, H, J)$ in the other direction one can show that $F \circ F'$ and $F' \circ F$ are chain-homotopic to the identity, so the induced continuation maps on Floer cohomology are inverses to each other.

Remark 3.37. For compact M, the extra term in the energy condition (11) is bounded. It turns out that for non-compact M we need the condition $\partial_s H(\tau(s), s, t) \geq 0$ which can be satisfied if and only if $H' \geq H$. Therefore, for non-compact M, continuation maps can only exist in one direction, confer [37, §1]. We will have to deal with this problem when introducing the wrapped Fukaya category.

3.6. **(Higher) product operations.** We now want to introduce product operations on the Floer complex to establish an A_∞ structure, following [8, §2.1 & §2.2]. For this we will make use of pseudo-holomorphic disks with several marked points and their moduli spaces as introduced in §3.1 & §3.3. From now on we always assume that Lagrangians are exact, graded, intersect transversely, do not bound a non-zero area (i.e. $[\omega] \cdot \pi_2(M, L) = 0$) and the moduli spaces have a natural orientation.

Definition 3.38. Let L_0, L_1 and L_2 be Lagrangian submanifolds of M. The *Floer product* is the $\mathbb{C}$-linear map $- \cdot - \colon CF(L_1, L_2) \otimes CF(L_0, L_1) \to CF(L_0, L_2)$ defined via

$$p_2 \cdot p_1 = \sum_{\substack{q \in \chi(L_0, L_2) \\ [u]:\operatorname{ind}([u])=0}} (\#\mathcal{M}(p_1, p_2, q, [u], J))q$$

where $p_1 \in \chi(L_0, L_1)$ and $p_2 \in \chi(L_1, L_2)$.

Note this definition makes sense as $\dim(\mathcal{M}(p_1, p_2, q, [u], J)) = \operatorname{ind}([u])$.

Remark 3.39. One has to be a little careful when considering the Hamiltonian perturbations to ensure compatibility (cf. [8, p. 18]); we omit the details here.

Proposition 3.40. *The Floer product satisfies the Leibniz rule (i.e. the A_∞-relation in degree two) with respect to the Floer differential*

$$\partial(p_2 \cdot p_1) = \pm(\partial p_2) \cdot p_1 \pm p_2 \cdot (\partial p_1).$$

The argument is similar to the cobordism argument in the proof of Proposition 3.34.

Proof. Let u be a pesudo-holomorphic disk with $\operatorname{ind}(u) = 1$. Then $\mathcal{M}(p_1, p_2, q, [u], J)$ is a one-dimensional space; it has a Gromov compactification $\overline{\mathcal{M}}(p_1, p_2, q, [u], J)$ whose boundary consists of broken strips (since we excluded any bubbling). Again, by Remark 3.20 the index of a strip bounded by two transverse Lagrangians is at least one; similarly, the index of a disk bounded by three transverse Lagrangians is at least zero (cf. Remark 3.28). Therefore, an index one disk in $\overline{\mathcal{M}}(p_1, p_2, q, [u], J)$ must break into an index one strip and an index zero disk at the boundary. There are precisely three such configurations possible:

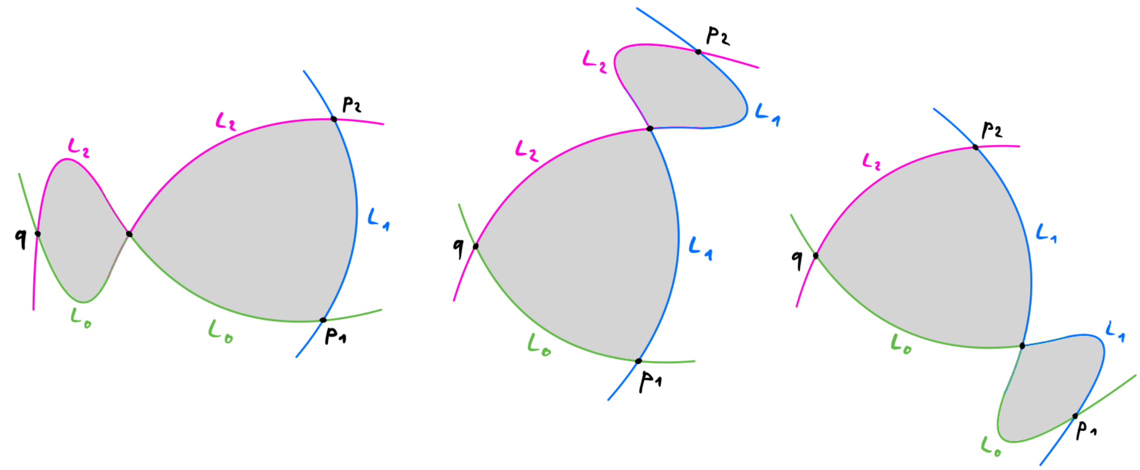


FIGURE 3.8. The three possible strip breakings on the boundary of $\overline{\mathcal{M}}(p_1, p_2, q, [u], J)$ contributing to the coefficient of q in $\partial(p_2 \cdot p_1)$, $(\partial p_2) \cdot p_1$ and $p_2 \cdot \partial(p_1)$, respectively

On the other hand it can be shown that any such configuration arises as the limit of a family of index one disks in $\overline{\mathcal{M}}(p_1, p_2, q, [u], J)$ (again cf. [29, App. A]). The three configurations above contribute to the coefficient of q in the expression for $\partial(p_2 \cdot p_1), (\partial p_2) \cdot p_1$ and $p_2 \cdot \partial(p_1)$, respectively. As $\overline{\mathcal{M}}(p_1, p_2, q, [u], J)$ is a compact manifold, the signed sum over the boundary computes to zero and we get the desired result. □

Corollary 3.41. *The Floer product induces a product on Floer cohomology $HF(L_1, L_2) \otimes HF(L_0, L_1) \to HF(L_0, L_2)$.* □

In a similar vein we can define higher product operations on the Floer complex

$$\mu^k \colon CF(L_{k-1}, L_k) \otimes \ldots \otimes CF(L_0, L_1) \to CF(L_0, L_k)[2-k],$$
$$\mu^k(p_k, \ldots, p_1) = \sum_{\substack{q \in \chi(L_0, L_k) \\ [u]: \mathrm{ind}([u]) = 2-k}} (\#\mathcal{M}(p_1, \ldots, p_k, q, [u], J))q. \tag{12}$$

To prove the algebraic relations between the μ^k it is then necessary to study the moduli space of stable pointed disks whose boundary consists of disks broken at at least one point, called the Stasheff associahedron (cf. [32, Lectures 9 & 10]). For the sake of brevity, we just state the following Proposition; a proof can be found in [32, Lecture 10].

Proposition 3.42. *The operations μ^k as defined above satisfy the A_∞-relations (1).*

3.7. **Summary of Floer cohomology.** To conclude the section on Floer cohomology we would like to give an overview of the conditions needed to make Floer cohomology well-defined and give a very simple first example:

(1) $2c_1(TM) = 0$ and $\mu_L = 0$ are necessary for the existence of graded Lagrangians to construct the grading of the Floer complex;

(2) L exact, so that the Gromov compactification $\overline{\mathcal{M}}(p, q; [u], J)$ exists (in general, the weaker "finite energy" condition is needed); exactness also enables us to work with $\mathbb{C}$-coefficients instead of Novikov-coefficients; moreover, exactness guarantees that $[\omega] \cdot \pi_2(M, L) = 0$, i.e. the symplectic area bounded by L vanishes; this is necessary to avoid disk and sphere bubbling on the boundary of $\overline{\mathcal{M}}(p, q; [u], J)$ which could lead to $\partial^2 \neq 0$;
(3) Pin-structure on L to make $\mathcal{M}(p, q; [u], J)$ orientable.

Example 3.43 ([8, Ex. 1.14]). Let S be the zero section of the cylinder T^*S^1 (with standard symplectic form $\omega = \mathrm{d}r \wedge \mathrm{d}\theta$)[3]. Clearly S is an exact Lagrangian submanifold. As a Hamiltonian perturbation we can pick $H(r, \theta) = \cos(\theta)$; then $\mathrm{d}H = -\sin(\theta)\mathrm{d}\theta$, so the Hamiltonian vector field is $X_H = -\sin(\theta)\frac{\partial}{\partial r}$ and its time one flow is $\phi_H^1(r, \theta) = (r - \sin(\theta), \theta)$. We see that S and $S' = \phi_H^1(S)$ have two intersection points, $p = (0, 0)$ and $q = (0, \pi)$. Moreover, we get two pseudo-holomorphic strips connecting p and q with boundary in S and S'.

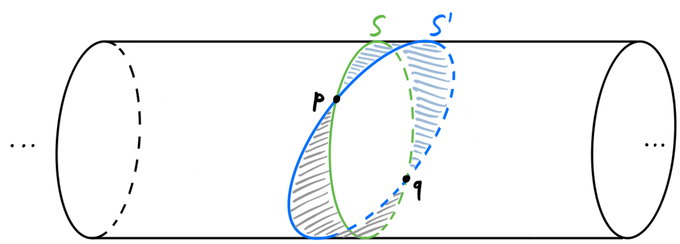


FIGURE 3.9. The zero section S of T^*S^1 with a Hamiltonian perturbation S'; the two shaded areas are the two pseudo-holomorphic strips connecting the intersection points

By Example 3.19, the strips have index one. This implies that the difference of the degrees of the two points is one (cf. Proposition 3.26); without loss of generality, $\deg(p) = 0$ and $\deg(q) = 1$. Therefore, the Floer complex is given by

$$CF(S, S) = \{\mathbb{C} \xrightarrow{\partial} \mathbb{C}\}$$

with the first $\mathbb{C}$ in degree zero. What is the differential ∂? Both strips contribute to the differential, but with opposite signs[4], so they cancel out and we get $\partial = 0$.

Let us consider the product μ^2 which has degree zero. We introduce a second Hamiltonian perturbation S'' which is similar to S' but with a small offset. To make the picture a bit clearer we view the cylinder unwrapped.

[3]You might wonder why we consider a non-compact manifold here as opposed to all considerations before; we don't have to worry about the non-compactness here as all the geometry takes place in a compact part S^1 times a closed interval. In the next chapter we will also deal with non-compact manifolds.

[4]Unfortunately, as we have not rigorously defined the signs, the reader has to believe this fact.

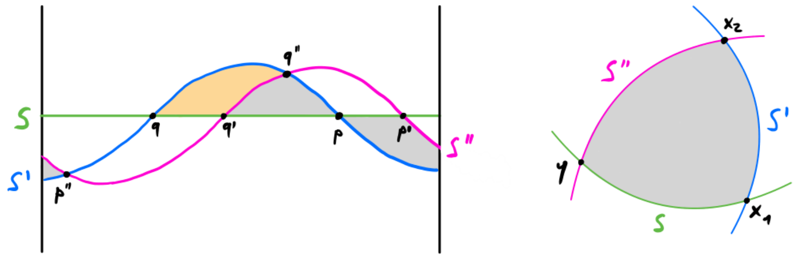


FIGURE 3.10. On the right side a pseudo-holomorphic disk contributing to the product $x_1 \cdot x_2 = y$; on the left the gray shaded triangles contribute to the product $p \cdot q = q$ and $p \cdot p = p$, respectively

The intersections between S and S'' are denoted by p' and q', corresponding to p and q; similarly, the intersections between S' and S'' are denoted by p'' and q''. The product is then a map

$$\mu^2: CF(S, S') \otimes CF(S', S'') \to CF(S, S'').$$

A pseudo-holomorphic disk contributing to the product $x_1 \cdot x_2 = y$ is depicted on the right-hand side of Figure 3.43; with this notation we are looking for triangles with y single-primed and x_2 double-primed. Indeed, we find two such triangles on the cylinder (gray shaded): they contribute to $p \cdot q'' = q'$ and $p \cdot p'' = p'$, i.e. $p \cdot q = q$ and $p \cdot p = p$, respectively. Therefore, p is the unit inside $CF(S, S)$ (in this case it is even a *strict* unit and not only a cohomological unit). Note that these are all possible product operations; in particular, the orange shaded triangle does not contribute to a product as it is not of the form depicted on the right (also note that the degrees do not match as $\deg(q) = 1$).

Recall that the higher product operations μ^k have degree $2 - k$, hence for $k > 2$ all operations vanish on $CF(S, S)$. Therefore, both the Floer complex $CF(S, S)$ as well as Floer cohomology $HF^*(S, S)$ can be identified with the ring of dual numbers, $\mathbb{C}[\epsilon]/(\epsilon^2)$ with $\deg(\epsilon) = 1$. In particular, we see that Floer cohomology $HF^*(S, S)$ is isomorphic to singular cohomology of the sphere, $H^*(S^1)$. This is an instance of a more general result which we won't prove here:

Proposition 3.44 ([8, Prop. 1.13]). *Under the assumptions listed at the beginning of §3.7, we have*

$$HF^*(L, L) \cong H^*(L, \mathbb{C}).$$

4. (Wrapped) Fukaya Categories

In this section we are going to define the object on one side of the homological mirror symmetry equivalence: the (wrapped) Fukaya category, a "categorification" of Floer cohomology.

4.1. **Fukaya categories.** Roughly speaking the Fukaya category of a symplectic manifold M consists of Lagrangian submanifolds of M as objects and Floer complexes as morphism spaces. As we have seen in the last chapter several assumptions are necessary to make the Floer complex and the higher product operations on it well-defined and we have to include these assumptions in the definition of the Fukaya category. In the following we will restrict, as done before, to the case of exact Lagrangians in an exact manifold which enables us to work with complex coefficients and which is sufficient for the case we are interested in.

Definition 4.1 ([8, cf. Def. 2.9]). Let (M, ω) we an exact symplectic manifold with $2c_1(TM) = 0$. The *Fukaya category* $\mathcal{F}(M)$ has as objects exact, compact, closed, oriented Lagrangian submanifolds $L \subset M$ bounding no symplectic area, $[\omega] \cdot \pi_2(M, L) = 0$, having vanishing Maslov class $\mu_L = 0$, and equipped with a grading and a Pin-structure. Moreover, for any pair of objects (L_0, L_1), choose perturbation data $H_{01} \in C^\infty([0,1] \times M, \mathbb{R})$ and $J_{01} \in C^\infty([0,1] \times M, \mathcal{J}(M, \omega)$ and for any tuple of objects $(L_0, \ldots, L_k)$ choose perturbation data H and J that are compatible with H_{ij} and J_{ij} (cf. Remark 3.39). Then the morphism spaces of $\mathcal{F}(M, \omega)$ are given by $\hom(L_0, L_1) = CF(L_0, L_1, H_{01}, J_{01})$, equipped with (higher) product operations as defined in (12). This makes $\mathcal{F}(M)$ into a non-unital but cohomologically unital A_∞-category over $\mathbb{C}$ (cf. Proposition 3.42).

As our main focus in this essay lies on the *wrapped* Fukaya category, we quickly move on.

4.2. **Wrapped Fukaya Categories.** The wrapped Fukaya category is the extension of the Fukaya category to non-compact manifolds. We first need to collect some notions from symplectic geometry.

Definition 4.2 ([28, Def. 3.5.27]). Let (M, ω) be a symplectic manifold. A vector field Z on M satisfying $\mathcal{L}_Z \omega = \omega$ is called *Liouville field*. For a Liouville field Z, the 1-form $\theta := \iota_Z \omega$ is called *Liouville form*.

The Liouville form satisfies $\mathrm{d}\theta = \mathrm{d}\iota_Z\omega = \mathrm{d}\iota_Z\omega + \iota\mathrm{d}\omega = \mathcal{L}_Z\omega = \omega$ by Cartan's magic formula.

Definition 4.3 ([28, Def. 3.5.32], [8, p. 32]). Let (M, ω) be a symplectic manifold containing a compact domain M^c with smooth boundary hypersurface ∂M^c. Suppose there is a global Liouville field Z on M with Liouville form θ whose restriction gives a contact form $\theta|_{\partial M^c}$ on ∂M^c and Z is positively transverse to ∂M^c and has no zeroes in $M \setminus M^c$. Then M is called a *Liouville manifold*.

Remark 4.4. Let ϕ_Z^t be the flow of Z. Then the map

$$(1, \infty) \times \partial M^c \to M \setminus M^c, \quad (r, p) \mapsto \phi_Z^{r-1}(p)$$

gives a symplectomorphism between $(M \setminus M^c, \omega)$ and $((1, \infty) \times \partial M^c, \mathrm{d}(r\theta|_{\partial M^c}))$. Under this identification the Liouville field becomes $Z = r\frac{\partial}{\partial r}$.

Example 4.5. Any cotangent bundle of a closed manifold is Liouville.

Definition 4.6. Let M be a Liouville manifold with Lioville form θ and Liouville flow ϕ^t. A submanifold $Y \subset M$ is said to be *conical at infinity* if there exists a compact subset $M^c \subset M$ such that $(\phi^t)_* TY_p = TY_{\phi^t(p)}$ for all $p \notin M^c$ and $\theta|_{Y \cap (M \setminus M^c)} = 0$.

Remark 4.7. Under the identification from Remark 4.4, a submanifold Y which is conical at infinity is either compact or corresponds to the cone $(1,\infty)\times(Y\cap\partial M^c)$ outside of M^c.

We are now readily prepared to give the definition of the wrapped Fukaya category.

Definition 4.8 ([8, §4.1])**.** Let $(M,\omega=\mathrm{d}\theta)$ be a Liouville manifold. The *wrapped Fukaya category* $\mathcal{W}(M)$ has as objects exact, graded Lagrangian submanifolds $L\subset M$ which are conical at infinity. The morphism spaces are given by *wrapped Floer complexes*: the definition is the same as for usual Floer complexes but we restrict the Hamiltonian perturbations to functions $H\colon M\to\mathbb{R}$ which on the non-compact part $(1,\infty)_r\times\partial M^c$ satisfy $H=r^2$ (which "wraps" the Lagrangians). Then for any pair of objects L_0,L_1 we have $\hom(L_0,L_1)=CW(L_0,L_1,H)$. The (higher) product operations defined in (12) make $\mathcal{W}(M)$ into an A_∞-category.

Remark 4.9. The last statement above requires some explanation: the product as defined in (12) gives a map

$$CW(L_1,L_2,H)\otimes CW(L_0,L_1,H)\to CW(L_0,L_2,2H).$$

What we really want is a map to $CW(L_0,L_2,H)$. The usual way to circumvent this issue by using a continuation map does not work in this case due to the non-compactness of M and the fact that $H\not\geq 2H$ (cf. Remark 3.37). We have to use the following rescaling trick to remedy this problem:

The Liouville flow in $(1,\infty)\times\partial M^c$ is given by $\phi_Z^t(r)=e^tr$. Let ψ^t be the time $\log t$ Liouville flow for $t>1$, i.e. $\psi^t(r)=tr$. There is a natural isomorphism

$$CW(L_0,L_1,H,J)\cong CW(\psi^t(L_0),\psi^t(L_1),t^{-1}H\circ\psi^t,\psi^t_*J);$$

in particular there is an isomorphism

$$CW(L_0,L_2,H,J)\cong CW(\psi^2(L_0),\psi^2(L_2),\frac{1}{2}H\circ\psi^2,\psi^2_*J). \tag{13}$$

Note that outside M^c, $\frac{1}{2}H\circ\psi^2=2H$ and therefore there exists a continuation map

$$CW(L_0,L_2,2H)\to CW(\psi^2(L_0),\psi^2(L_2),\frac{1}{2}H\circ\psi^2)$$

(as on the compact part continuation maps always exist). For details the reader is referred to [3, §3.2].

Note that $\mathcal{W}(M)$ contains $\mathcal{F}(M^c)$ as a full A_∞-subcategory as compact Lagrangians are not influenced by the wrapping at infinity. The first example of a wrapped Fukaya category is presented in the next section.

5. Homological Mirror Symmetry for $\mathbb{C}^*$

In this chapter the main theorem of this essay is proven.

Theorem 5.1. *There is an equivalence of triangulated categories between* $D\mathcal{W}(\mathbb{C}^*)$ *and* $D^b\mathrm{Coh}(\mathbb{C}^*)$ *where* $\mathbb{C}^*$ *is considered as* T^*S^1 *and* $\mathrm{Spec}(\mathbb{C}[x,x^{-1}])$*, respectively.*

To do so we will first calculate parts of the wrapped Fukaya category of $\mathbb{C}^*$. A theorem by Abouzaid which we will not prove states that $D\mathcal{W}(\mathbb{C}^*)$ is generated by a single cotangent fibre. We will use this statement to prove Theorem 5.1 and then explore some manifestations of the established duality.

5.1. **Wrapped Fukaya Category of $\mathbb{C}^*$.** We consider $M = \mathbb{C}^*$ as $T^*S^1 = \mathbb{R}_r \times S^1_\theta$ with Liouville form $\lambda = r d\theta$ (so $\omega = \mathrm{d}r \wedge \mathrm{d}\theta$) and Hamiltonian perturbation $H(r, \theta) = r^2$. Let $L = \mathbb{R} \times \{0\}$ be a Lagrangian cotangent fibre. Indeed, λ vanishes when restricted to L and L is invariant under the Liouville flow $\phi^t_Z(r, \theta) = (e^t r, \theta)$ so $L \in \mathrm{Ob}(\mathcal{W}(M))$. We want to calculate $CW(L, L)$.

As already seen in Example 3.33, $\phi^1_H(L)$ wraps around the cylinder. The intersections points of $\phi^1_H(L)$ with L are the generators of the Floer complex; we can identify them as $\chi(L, L) = \{x_i \mid i \in \mathbb{Z}\}$ where each x_i lies on the line $\{\theta = 0\}$ in increasing order with x_0 lying on the zero section. Every point x_i has the same degree as in a fixed trivialisation the pairs of spaces $T_{x_i}L$ and $T_{x_i}\phi^1_H(L)$ agree for all $i \in \mathbb{Z}$; without loss of generality we can then assume $\deg(x_i) = 0$ for all $x_i \in \chi(L, L)$.

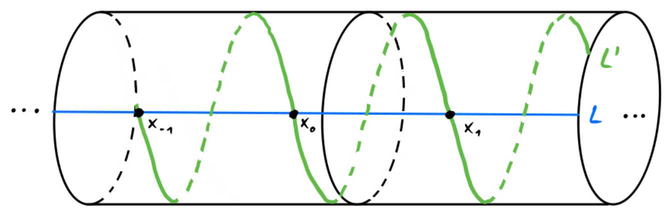


FIGURE 5.1. The wrapped Lagrangian L' intersecting the cotangent fibre L in points x_i

We can also view the wrapping inside the universal cover of $\mathbb{R} \times S^1$ to make the picture a bit more tidy where we use the notation $L' = \phi^1_H(L)$:

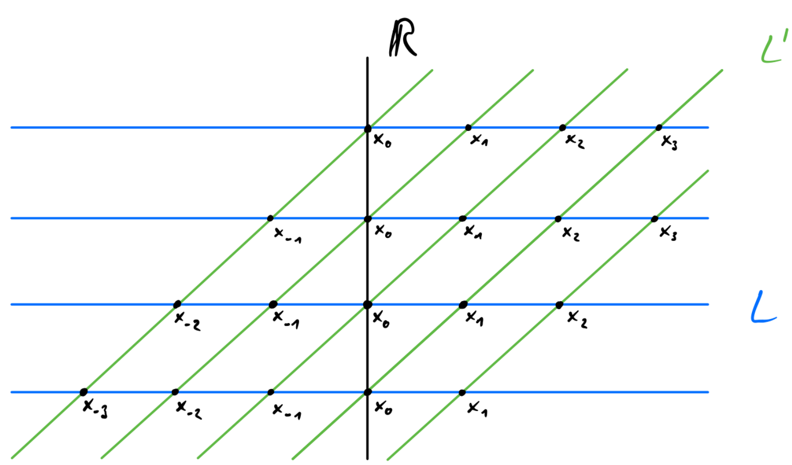


FIGURE 5.2. The intersection of L with L' inside the universal cover $\mathbb{R}^2$ of T^*S^1

To determine the (higher) product operations μ^k, recall that μ^k has degree $2-k$; as all generators have the same degree, the only possibly non-vanishing operation is the product μ^2 which we explore now. The rescaling trick described in Remark 4.9 involves the logarithmic time Liouville flow $\psi^2\colon (r,\theta) \mapsto (2r,\theta)$. But L is completely invariant under ψ^t and $\frac{1}{2}H \circ \psi^2 = 2H$ globally, so the product is a map

$$CW(L,L,H) \otimes CW(L,L,H) \to CW(L,L,2H)$$

counting pseudo-holomorphic strips of index zero with boundary in $L, L' = \phi_H^1(L)$ and $L'' = \phi_H^2(L)$. Suppose we want to determine the coefficient of $q \in \chi(L,L)$ in the product $p_1 \cdot p_2$. We denote the generators of $CW(L,L,2H)$ by $\{y_i \mid i \in \mathbb{Z}\}$; as ϕ_H^2 wraps twice as fast around L as ϕ_H^1, the point x_i corresponds to y_{2i}. Moreover, we view p_2 as an element of $L' \cap L$ and $\phi_H^1(p_1)$ as an element of $L'' \cap L'$. Again, we take a look at the universal cover:

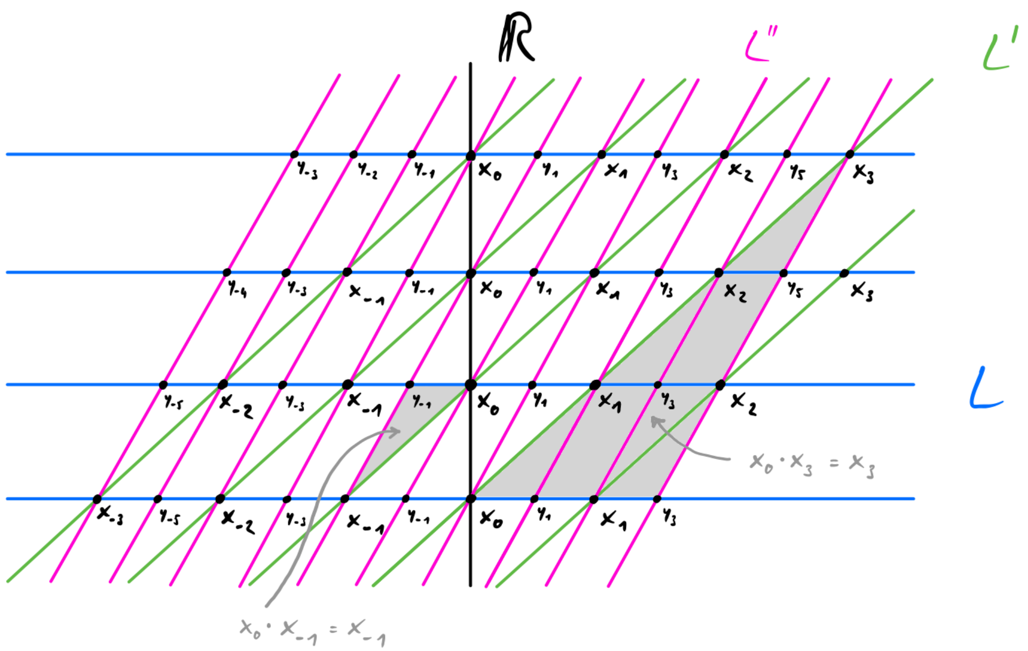


FIGURE 5.3. The cotangent fibre L intersecting with its perturbed versions L' and L''; the shaded triangles are pseudo-holomorphic disks contributing to the coefficient of x_{-1} in $x_0 \cdot x_{-1}$ (small left triangle) and to the coefficient of x_3 in $x_0 \cdot x_3$ (big right triangle)

Suppose p_2 corresponds to x_i and p_1 to x_j. Then there is a unique triangle with two vertices being x_i, x_j and edges being L, l' and L''; the third vertex of this triangle is clearly y_{i+j}. As all points have degree zero, this triangle has Maslov index zero, therefore it contributes to the product map. Under the rescaling isomorphism (13) we then obtain the relation

$$x_j \cdot x_i = x_{i+j}.$$

Another way to see the vanishing of all other μ^k, $k \neq 2$ would be by realising that there don't exist any pseudo-holomorphic polygons on the cylinder with the appropriate boundary. Therefore, we get an isomorphism of A_∞-algebras:

$$CW(L, L) \cong \mathbb{C}\Big[x, x^{-1}\Big], \quad x_i \mapsto x^i.$$

We will explore more of $\mathcal{W}(T^*S^1)$ in section 5.3 but for now we move on to the generation.

To establish the homological mirror symmetry equivalence we will use the following result by Abouzaid [4]. A proof of this result is beyond the scope of this essay.

Theorem 5.2. *If Q is an oriented closed smooth manifold, then any cotangent fibre generates $\mathcal{W}(T^*Q)$.*

Note that the derived category Tw$\mathcal{W}(M)$ is generated by $\mathcal{W}(M)$, so any cotangent fibre also generates Tw$\mathcal{W}(T^*Q)$. Recall that in our case of T^*S^1, $CW(L, L)$ is concentrated in degree zero. Therefore, L generates $D\mathcal{W}(T^*S^1) = H^0(\mathrm{Tw}\mathcal{W}(T^*S^1))$ as a triangulated category.

5.2. **Derived Category of $\mathbb{C}^*$.** The way one usually introduces the bounded derived category of coherent sheaves is via localising at quasi-isomorphisms in the homotopy category of bounded complexes. But there is a different way we just briefly sketch here [11, §3.2 & §3.3]: by replacing modules with their resolutions we get graded Hom-spaces where each component is given by an Ext-group. The Ext^0-group recovers the usual Hom-space. We can equip the Ext-groups with a dg-algebra structure induced by composition of morphisms; in that way we get an A_∞-structure on the module category. It makes then sense to extend to its twisted category and take the zero cohomology part, analogously to Definition 2.15. This is an alternative way to construct D^bCoh.

As a side remark, the derived category of a scheme contains a lot of information about the geometry of the scheme. In fact, in many cases the scheme can be recovered from its derived category which is a result by Bondal and Orlov.

Theorem 5.3 ([12, Thm. 2.5])**.** *Let X be a smooth irreducible projective variety with ample canonical or anticanonical sheaf. If $D^b\mathrm{Coh}(X)$ is equivalent as a graded category to $D^b\mathrm{Coh}(X')$ for some smooth algebraic variety X', then X is isomorphic to X'.*

The general idea of the proof is that the skyscraper sheaves contain the information about the closed points of X and one can reconstruct the Zariski topology from line bundles.

Now we come back to our example and interpret $\mathbb{C}^*$ as $\mathrm{Spec}(\mathbb{C}[x, x^{-1}])$. Note that $\mathbb{C}[x, x^{-1}]$ is a principal ideal domain (clearly $\mathbb{C}[x]$ is a PID, $\mathbb{C}[x, x^{-1}] = S^{-1}\mathbb{C}[x]$ for $S = \{x^i \mid i \in \mathbb{N}\}$ and localizations of PIDs are again PIDs). By the structure theorem for finitely generated modules over a PID, all finitely generated modules M over $A :=$ $\mathbb{C}[x, x^{-1}]$ are of the form $M = A/(a_1) \oplus \ldots \oplus A/(a_n)$; in particular, every finitely generated module over $\mathbb{C}[x, x^{-1}]$ is also finitely presented and has a two-term resolution

$$0 \to A^n \xrightarrow{(a_1, \ldots, a_n)} A^n \to M \to 0.$$

In other words, the derived category $D^b\mathrm{MOD}(A)$ is equivalent to its triangulated subcategory $\mathrm{Perf}(A)$ consisting of modules with finite resolutions (perfect objects). Therefore, A is a generator of $D^b\mathrm{MOD}(A)$, and as $D^b\mathrm{MOD}(A)$ is equivalent to $D^b\mathrm{Coh}(\mathbb{C}^*)$, we conclude that the structure sheaf $\mathcal{O}_{\mathbb{C}^*}$ generates $D^b\mathrm{Coh}(\mathbb{C}^*)$ (by taking mapping cones and applying shifts).

Example 5.4. Let us give a brief sketch how one can build a resolution via mapping cones in $D^b\mathrm{Coh}$. Suppose we have a two-term resolution

$$0 \to F \to G \to H \to 0$$

corresponding to a class $\sigma \in \mathrm{Ext}^1(H, F) \cong \hom(H[-1], F)$, so σ gives a morphism

$$f\colon H[-1] \to F$$

and by taking its mapping cone we obtain an exact triangle

$$H[-1] \to F \to C(f) \to H$$

which is isomorphic to

$$F \to G \to H \to F[1]$$

by rotation and the 2-out-of-3-property. Longer resolutions can be similarly obtained by taking repeated mapping cones.

5.3. **Homological Mirror Symmetry.** Finally we are able to prove Theorem 5.1.

Proof of Thm. 5.1. By Theorem 5.2, the cotangent fibre L generates $\mathcal{W}(T^*S^1)$. We have

$$\hom_{\mathcal{W}(T^*S^1)}(L, L) \cong \mathbb{C}\left[x, x^{-1}\right] := A.$$

The Yoneda embedding

$$\begin{array}{rcl} \mathcal{Y}\colon \mathcal{W}(T^*S^1) & \to & \mathrm{MOD}(A) \\ X & \mapsto & \hom_{\mathcal{W}(T^*S^1)}(L, X) \end{array}$$

gives a cohomologically fully faithful quasi-embedding $\mathcal{W}(T^*S^1) \hookrightarrow \mathrm{MOD}(A)$ [34, I §2 & §3]. As $\mathrm{MOD}(A)$ is triangulated and $\mathrm{Tw}\mathcal{W}(T^*S^1)$ is a triangulated envelope of $\mathcal{W}(T^*S^1)$, the Yoneda functor extends to a cohomologically fully faithful quasi-embedding $\widetilde{\mathcal{Y}}\colon \mathrm{Tw}\mathcal{W}(T^*S^1) \hookrightarrow \mathrm{MOD}(A)$, confer [34, I Lem. 3.36]. The image of $\widetilde{\mathcal{Y}}$ consists precisely of modules with finite resolutions, confer [34, I Cor. 5.26]. By taking zeroth cohomology we obtain an equivalence

$$D\mathcal{W}(T^*S^1) \cong \mathrm{Perf}(A).$$

As we have seen above, $\mathrm{Perf}(A) \cong D^b\mathrm{Coh}(\mathbb{C}^*)$ which concludes the argument. □

As we have found that the structure sheaf $\mathcal{O}_{\mathbb{C}^*}$ corresponds to a cotangent fibre L under mirror symmetry, we might ask what the mirror L_p of a skyscraper sheaf $\mathcal{O}_p$ is. First note that $\hom_{D^b\mathrm{Coh}(\mathbb{C}^*)}(\mathcal{O}_{\mathbb{C}^*}, \mathcal{O}_p) \cong \mathbb{C}$, so we expect $L \cap L_p$ to consist of a single point. A good candidate for L_p would therefore be the zero section S inside T^*S^1 which we prove in the following. Before doing this it might be enlightening to look at $CW(L, S)$ and $CW(S, L)$.

Both complexes are generated by a single point, the unique point p where S and L intersect transversely. Moreover, one can see that there is no disk with boundary in S and L connecting p to itself, so all μ^k vanish and both complexes are isomorphic to $\mathbb{C}$ concentrated in a single degree.

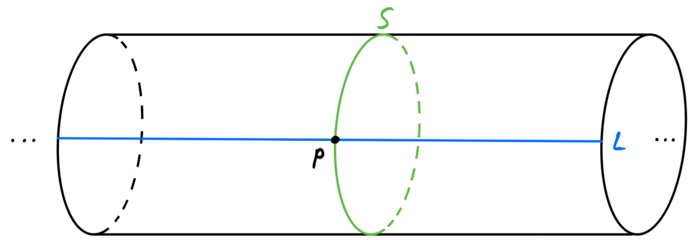


FIGURE 5.4. The cotangent fibre L intersecting the zero section S in a single point

However, we know that $\hom_{D^b\mathrm{Coh}(\mathbb{C}^*)}(\mathcal{O}_{\mathbb{C}^*}, \mathcal{O}_p) \cong \mathbb{C}$ whereas $\hom_{D^b\mathrm{Coh}(\mathbb{C}^*)}(\mathcal{O}_p, \mathcal{O}_{\mathbb{C}^*}) = 0$: replacing $\mathcal{O}_p$ by a resolution, we see that a morphism from $\mathcal{O}_p$ to $\mathcal{O}_{\mathbb{C}^*}$ would be given by two maps f_0, f_1 making the following diagram commute:

$$\begin{tikzcd} \mathcal{O}_{\mathbb{C}^*} \arrow[r, "s_p"] \arrow[d, "f_1"'] & \mathcal{O}_{\mathbb{C}^*} \arrow[d, "f_0"] \\ 0 \arrow[r] & \mathcal{O}_{\mathbb{C}^*} \end{tikzcd}$$

But only the zero maps fit into this diagram. Where does this discrepancy come from? Let us look at morphisms between $\mathcal{O}_p$ and $\mathcal{O}_{\mathbb{C}^*}[1]$; they are given by maps between complexes

$$\begin{tikzcd} \mathcal{O}_{\mathbb{C}^*} \arrow[r, "s_p"] \arrow[d, "f_1"'] & \mathcal{O}_{\mathbb{C}^*} \arrow[d, "f_0"] \\ \mathcal{O}_{\mathbb{C}^*} \arrow[r] & 0 \end{tikzcd}$$

and we see that f_1 can be multiplication by an arbitrary complex number. Therefore, we get $\mathrm{Ext}^1(\mathcal{O}_p, \mathcal{O}_{\mathbb{C}^*}) \cong \mathbb{C}$, so there is just a shift in the degrees. Indeed, we can also see this degree shift between $CW(L, S)$ and $CW(S, L)$: recall that the degree of p is calculated by taking the winding number of the concatenation of the reversed canonical short path between T_pL and T_pS (i.e. anti-clockwise rotation) with the unique path obtained from the lifting into $\widetilde{\mathrm{LGr}}(T_p(T^*S^1))$. For T_pL and T_pS, the former path is always rotation by $\pi/2$; however, the latter path changes direction: in $CW(S, L)$ it is rotation by $\pi/2$, in $CW(L, S)$ it is rotation by $-\pi/2$. Therefore, in one case they concatenate to a closed loop in $\widetilde{\mathrm{LGr}}(T_p(T^*S^1))$, in the other case they just cancel out. Hence, p has degree one in $CW(S, L)$ and degree zero in $CW(L, S)$ which is exactly what we expect.

Now we prove the mirror correspondence between $\mathcal{O}_p$ and S by considering how they emerge from $\mathcal{O}_{\mathbb{C}^*}$ and L, respectively. The coordinate of p defines a section s_p of $\mathcal{O}_{\mathbb{C}^*}$ giving rise to an exact triangle

$$\mathcal{O}_{\mathbb{C}^*} \xrightarrow{s_p} \mathcal{O}_{\mathbb{C}^*} \xrightarrow{q_p} \mathcal{O}_p \to \mathcal{O}_{\mathbb{C}^*}[1]. \tag{14}$$

Hence, in $D^b\mathrm{Coh}(\mathbb{C}^*)$ the skyscraper sheaf $\mathcal{O}_p$ is isomorphic to the mapping cone $C(s)$. It turns out that mapping cones in the Fukaya category of T^*S^1 are related to Dehn twists.

Definition 5.5. Let S denote the zero section inside T^*S^1 and let $U \cong S \times [0,1]$ be a tubular neighbourhood around S with coordinates $(\theta, t) \in [0, 2\pi] \times [0,1]$. The *Dehn twist* τ_S about S is defined to be the identity map in the complement of U and inside U via

$$\tau_S(\theta, t) = (e^{i(\theta + 2\pi t)}, t).$$

One can also define the Dehn twist for higher dimensional spheres. By Weinstein's tubular neighbourhood Theorem, any Lagrangian sphere S in a symplectic manifold M has a neighbourhood symplectomorphic to a neighbourhood of the zero section inside T^*S^n; therefore, one can define the Dehn twist for arbitrary Lagrangian spheres, giving a symplectomorphism of T^*S^n [8, §3.3].

Example 5.6. The Dehn twist of a cotangent fibre L inside T^*S^1 about the zero section S gives a Lagrangian which is Hamiltonian isotopic to L again.

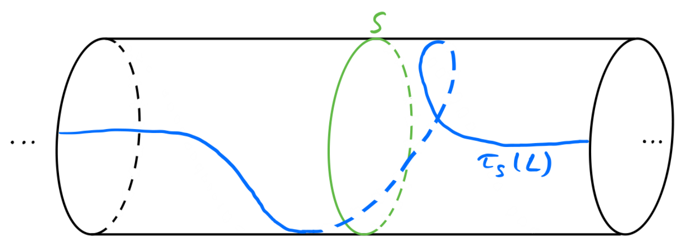


FIGURE 5.5. The Dehn twist $\tau_S(L)$ of the cotangent fibre L about the zero section S

A theorem by Seidel tells us how the Dehn twist relates to L and S.

Theorem 5.7 ([8, Thm. 3.8]). *Let (M, ω) be a compact symplectic manifold, let S be a Lagrangian sphere and let $L \in \mathrm{Ob}(\mathcal{F}(M))$. Then there is an exact triangle in* $\mathrm{Tw}\mathcal{F}(M)$

$$HF^*(S, L) \otimes S \xrightarrow{ev} L \to \tau_S(L) \to (HF^*(S, L) \otimes S)[1]$$

where ev is the canonical evaluation map defined as follows: let $\{p_i\}_{i \in I}$ be a set of generators for $HF^*(S, L) = H^*\mathrm{hom}(S, L)$, so each p_i is represented by a morphism $S \to L$ which we also denote by p_i. Then $HF^*(S, L) \otimes S = \bigoplus_{i \in I} S \cdot p_i$ and ev is given by mapping the summand $S \cdot p_i$ to L via p_i.

Now let S be the zero section of T^*S^1. Note the Dehn twist only affects the compact part of M and $\mathcal{W}(M)$ contains $\mathcal{F}(M^c)$ as a full subcategory. Then applying Theorem 5.7 and using $HF^*(S, L) \cong \mathbb{C} \cdot p \cong \mathbb{C}[-1]$ (as $\deg(p) = 1$), we get an exact triangle in $D\mathcal{W}(T^*S^1)$

$$S[-1] \xrightarrow{ev_p} L \to L \to S.$$

Comparing this with the exact triangle (14) we deduce that $\mathcal{O}_p$ corresponds to S under the mirror.[5] Note that as a nice consequence we can easily calculate $\mathrm{Ext}^1(\mathcal{O}_p, \mathcal{O}_p)$ as the degree one part of $CF(S, S)$ which is $\mathbb{C}$.

[5]This also legitimates the slightly sloppy doubled notation of p as a point in $\mathbb{C}^*$ and as a generator of $HF^*(S, L)$.

By now we have identified a the skyscraper sheaf of a single point with the zero section. What are the mirrors of the other skyscraper sheaves? To answer this question we enrich the Fukaya category by *local systems*. The reason why we don't see mirrors of other skyscraper sheaves right now is that they correspond to non-exact Lagrangians (one could think of them as translations of S along the cylinder). One would be able to see them if we worked with Novikov-coefficients.

Definition 5.8. Let L be a Lagrangian submanifold. A *rank one local system* on L is a complex line bundle $\mathcal{L} \to L$ equipped with a flat connection ∇.

Remark 5.9 ([14, §1]). Equivalently, one can think of a rank one local system as a locally constant sheaf on L with stalks being $\mathbb{C}$. If L is path-connected, rank one local systems are in a bijective correspondence with representations

$$\rho\colon \pi_1(L) \to \mathbb{C}^*.$$

The definition of the Floer complex and the (higher) product operations need to be adapted as follows, confer [8, Rem. 2.11]. The new Floer complex is given by

$$CF((L_0, \mathcal{L}_0), (L_1, \mathcal{L}_1)) \coloneqq \bigoplus_{p \in \chi(L_0, L_1)} \hom(\mathcal{L}_0|_p, \mathcal{L}_1|_p).$$

To redefine the operation μ^k, consider objects $(L_0, \mathcal{L}_0), \dots, (L_k, \mathcal{L}_k)$ with pairwise intersections $p_1, \dots, p_k, q$; to simplify the notation we additionally set $p_0 = p_{k+1} = q$. Let u be a pseudo-holomorphic disk with boundary in $L_0, \dots, L_k$ and marked points $p_1, \dots, p_k, q$. Let $\gamma_i \in \hom(\mathcal{L}_i|_{p_i}, \mathcal{L}_i|_{p_{i+1}})$ be the morphism given by parallel transport along the boundary of u lying on L_i. The new "points" in the Floer complex are morphisms $\rho_i \in \hom(\mathcal{L}_{i-1}|_{p_i}, \mathcal{L}_i|_{p_i})$. The composition $\gamma_k \circ \rho_k \circ \dots \circ \gamma_1 \circ \rho_1 \circ \gamma_0$ gives an element of $\hom(\mathcal{L}_0|_q, \mathcal{L}_k|_q)$. The μ^k are redefined as

$$\begin{aligned} \mu^k\colon & CF((L_{k-1}, \mathcal{L}_{k-1}), (L_k, \mathcal{L}_k)) \otimes \dots \otimes CF((L_0, \mathcal{L}_0), (L_1, \mathcal{L}_1)) \\ & \to CF((L_0, \mathcal{L}_0), (L_k, \mathcal{L}_k))[2-k], \end{aligned}$$

$$\mu^k(\rho_k, \dots, \rho_1) = \sum_{\substack{q \in \chi(L_0, L_k) \\ [u]: \mathrm{ind}([u]) = 2-k}} (\#\mathcal{M}(p_1, \dots, p_k, q, [u], J))(\gamma_k \circ \rho_k \circ \dots \circ \gamma_1 \circ \rho_1 \circ \gamma_0).$$

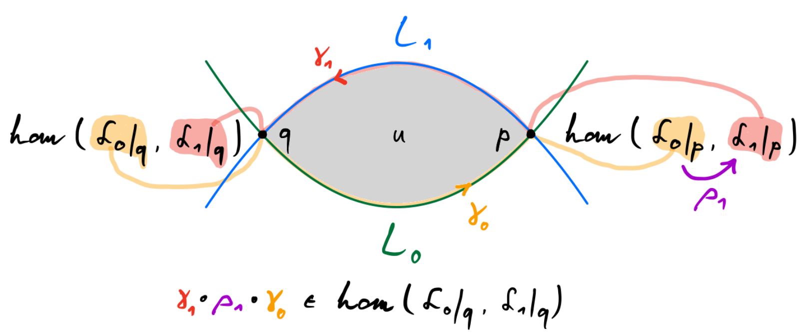


FIGURE 5.6. Illustration how a pseudo-holomorphic strip contributes to the redefined Floer differential, i.e. the operation μ^1

Let us come back to the example of T^*S^1. First note that the cotangent fibre L is contractible, so introducing local systems has no effect on the Floer complex involving L. However, if we consider the zero section S, by Remark 5.9 local systems on S are in one-to-one correspondence with points in $\mathbb{C}^*$. Let $z \in \mathbb{C}^*$ and let S_z be S equipped with the local system corresponding to z, then S_z is mirror dual to the skyscraper sheaf $\mathcal{O}_z$.

Let us study the morphisms $\hom(S_{z_1}, S_{z_2})$; we do this similarly to Example 3.43 where we now consider S_{z_1} instead of S and S_{z_2} instead of S'. The Floer complex is given by

$$\hom(\mathcal{L}_1|_p, \mathcal{L}_2|_p) \xrightarrow{\partial=0} \hom(\mathcal{L}_1|_q, \mathcal{L}_2|_q)$$

in degrees zero and one, for some locally constant sheaves $\mathcal{L}_1$ and $\mathcal{L}_2$. But if $z_1 \neq z_2$, there will be no morphisms between $\mathcal{L}_1$ and $\mathcal{L}_2$. Therefore, we conclude

$$CF(S_{z_1}, S_{z_2}) \cong \begin{cases} \mathbb{C}[\epsilon]/(\epsilon^2) & \text{if } z_1 = z_2 \\ 0 & \text{else} \end{cases}$$

in agreement with

$$\operatorname{Ext}^*(\mathcal{O}_{z_1}, \mathcal{O}_{z_2}) \cong \begin{cases} \mathbb{C}[\epsilon]/(\epsilon^2) & \text{if } z_1 = z_2 \\ 0 & \text{else.} \end{cases}$$

6. Partially Wrapped Fukaya Categories

In this last section we want to briefly sketch (without giving rigorous proofs) how to obtain mirror symmetry for the affine line $\mathbb{A}^1$ and the projective space $\mathbb{P}^1$ via partially wrapped Fukaya categories.

6.1. **Partially wrapped Fukaya categories.** Let us first take a look at the algebro-geometric side: $\mathbb{A}^1 = \operatorname{Spec}(\mathbb{C}[x])$ and $D^b\mathrm{Coh}(\mathbb{A}^1)$ consists of finitely generated modules over $\mathbb{C}[x]$. In particular, the derived category is again generated by the structure sheaf $\mathcal{O}_{\mathbb{A}^1}$ but with the difference that now

$$\hom_{D^b\mathrm{Coh}(\mathbb{A}^1)}(\mathcal{O}_{\mathbb{A}^1}, \mathcal{O}_{\mathbb{A}^1}) = \mathbb{C}[x].$$

We still have $\operatorname{Ext}^*(\mathcal{O}_z, \mathcal{O}_{\mathbb{A}^1}) = \mathbb{C}[-1]$, $\operatorname{Ext}^*(\mathcal{O}_{\mathbb{A}^1}, \mathcal{O}_z) = \mathbb{C}$

$$\operatorname{Ext}^*(\mathcal{O}_{z_1}, \mathcal{O}_{z_2}) \cong \begin{cases} \mathbb{C}[\epsilon]/(\epsilon^2) & \text{if } z_1 = z_2 \\ 0 & \text{else.} \end{cases}$$

Recalling that under mirror symmetry of $\mathbb{C}^*$, the negative powers of x in $\mathbb{C}[x, x^{-1}] = \hom_{D^b\mathrm{Coh}(\mathbb{C}^*)}(\mathcal{O}_{\mathbb{C}^*}, \mathcal{O}_{\mathbb{C}^*})$ corresponded to the intersections in the negative infinity direction of the cotangent fibre L on the cylinder with a wrapped cotangent fibre, we see that we somehow have to stop the wrapping towards negative infinity when we want to obtain the mirror of $\mathbb{A}^1$. This is achieved by adding a *stop* to the cylinder.

Definition 6.1 ([11, Def. 6.49]). Let (M, θ) be a Liouville manifold with compact domain M^c and conical end $(1, \infty)_r \times \partial M^c$ at infinity. Let U^c be a hypersurface in ∂M^c and let $U = (1, \infty) \times U^c$. Consider a Hamiltonian $H\colon M \to \mathbb{R}$ satisfying $H = r^2$ on the non-compact part (precisely as in the definition of the wrapped Fukaya category). If the Hamiltonian flow is everywhere normal to U we call U a *stop*.

Example 6.2. In the case of the cylinder T^*S^1 we can think of a stop as a set of points added to the "boundary" of the cylinder at infinity.

We now give a sketchy definition of the partially wrapped Fukaya category, confer [11, Def. 6.51].

Definition 6.3. Let (M, θ) be a Liouville manifold and let σ be a set of stops on M. The *partially wrapped Fukaya category* $\mathcal{W}_\sigma(M)$ is defined precisely as $\mathcal{W}(M)$ but with the restriction that towards infinity, the Hamiltonian wrapping goes as far as possible without crossing σ.

Let us now add a single stopping point to the cylinder and consider the cotangent fibre L as in section 5.1. Again we consider the wrapping $\phi_H^1(L)$—but this time the stop prevents L to wrap around the cylinder in the direction of negative infinity. Therefore, the remaining intersection points are $\chi(L, L) = \{x_i \mid i \in \mathbb{Z}_{\geq 0}\}$.

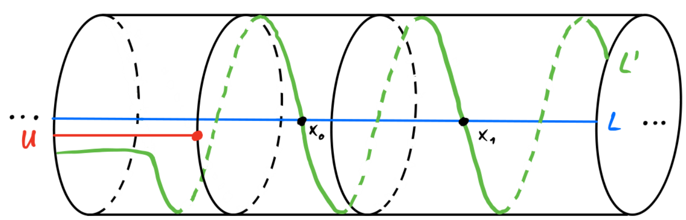


FIGURE 6.1. A stopping point U on the cylinder preventing the perturbed cotangent fibre L' to continue wrapping in the left direction

Clearly, all relations remain the same, i.e. we still have the product rule $x_j \cdot x_i = x_{i+j}$ and all x_i sitting in degree zero; thus $CW_\sigma(L, L) \cong \mathbb{C}[x]$. Moreover, the added stop does not affect the zero section S as it is in the compact part of the cylinder, so all relations involving S remain the same as in $\mathcal{W}(T^*S^1)$. We see that all morphism spaces in $\mathcal{W}_\sigma(T^*S^1)$ are isomorphic to the ones in $D^b\mathrm{Coh}(\mathbb{A}^1)$ at the beginning of this section, suggesting that the mirror to $\mathbb{A}^1$ is $\mathbb{C}^*$ with an additional stopping point. Notice that compared to mirror symmetry for $\mathbb{C}^*$ we have added in some way a single point on both sides (when we identify $\mathbb{A}^1$ with $\mathbb{C}$).

In $D^b\mathrm{Coh}(\mathbb{A}^1)$ there is also an object which does not occur in $D^b\mathrm{Coh}(\mathbb{C}^*)$, namely the skyscraper sheaf at the origin (the point we added) $\mathcal{O}_0$. Can we find its mirror object? Indeed, also in $\mathcal{W}_\sigma(T^*S^1)$ there is a new object: the Lagrangian, let us call it P, which starts and ends on the same side of the cylinder and turns around the stopping point.

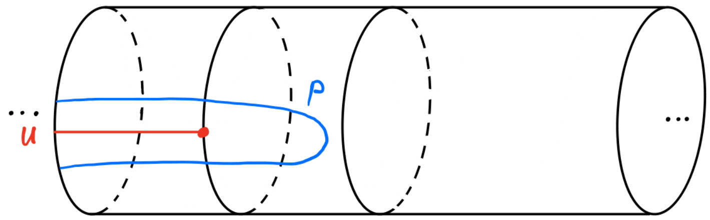


FIGURE 6.2. A new Lagrangian P corresponding to the skyscraper sheaf $\mathcal{O}_0$

In the wrapped Fukaya category without a stop this Lagrangian is isomorphic to the zero object. One way to see this is by noticing that the identity map id_P is in the image of the Floer differential [11, Ex. 6.47].

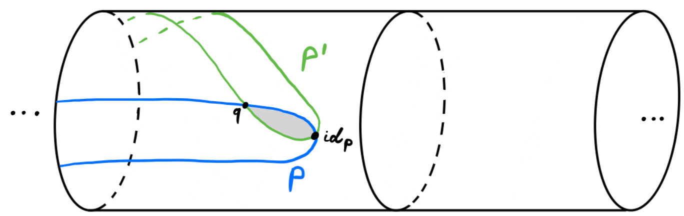


FIGURE 6.3. The Lagrangian P together with a Hamiltonian perturbation P'; one can find that id_P is in the image of the differential, $\mathrm{id}_P = \partial q$

Therefore, the Floer cohomology class of the identity map is zero which means that P itself must be isomorphic to the zero object inside $D\mathcal{W}(T^*S^1)$. The kind of perturbation depicted above is no longer valid once we added a stopping point as in Figure 6.1.

It turns out that in the case of (real) surfaces there is an entirely combinatorial description of the partially wrapped Fukaya category, introduced by Bocklandt [10] and Haiden, Katzarkov and Kontsevich [22]. In the following we give a brief sketch of the construction.

6.2. Marked surfaces.

Definition 6.4 ([22, §3.2]). Let S be a surface with corners, let $\partial_0 S$ and $\partial_1 S$ denote the zero- and one-dimensional boundary, respectively. Let $m \subset \partial S$ be the closure of a collection of components of $\partial_1 S$ such that the boundary of m in ∂S is precisely $\partial_0 S$. The pair (S, m) is called a *marked surface*.

The general idea is that the union of open intervals around stopping points on the boundary of S corresponds to $\partial S \setminus m$.

Definition 6.5 ([22, §3.2]). Let (S, m) be a marked surface. An *arc* in (S, m) is an embedded closed interval which intersects m transversely in its endpoints and which is not isotopic to an interval embedded in m. Here, isotopy means an isotopy keeping endpoints inside m. A *boundary arc* is an arc which is isotopic to the closure of a component of $\partial S \setminus m$. An *arc system* is a collection of pairwise disjoint and non-isotopic arcs.

Arcs will correspond to Lagrangian submanifolds of S and boundary arcs are the Lagrangians going around a stop as in Figure ???. One can equip the arcs with a grading, but we omit the details here, see [22, §2.1].

Definition 6.6 ([22, §3.3]). Let (S, m) be a marked surface. A *boundary path* is a non-constant path in m with orientation opposite to the orientation of the boundary.

It is possible to associate to a graded marked surface (S, m) and a graded arc system A a strictly unital A_∞-category $\mathcal{F}_A(S)$ with $\mathrm{Ob}(\mathcal{F}_A(S)) = A$ and where the boundary paths with endpoints in X and Y form a basis of $\hom_{\mathcal{F}_A(S)}(X, Y)$. The product structure is given by composition (if possible) of boundary paths. [22, §3.3]. The category $\mathcal{F}_A(S)$ can be very explicitly described.

Proposition 6.7 (cf. [22, §3.4] & [27, §2]). *Let (S, m) be a marked surface and let $A = \{a_i\}_i$ be an arc system. If $S \setminus \{\coprod_i a_i\}$ is a union of disks each of which has exactly one component of $\partial S \setminus m$ on its boundary, then the $\{a_i\}_i$ generate $D\mathcal{F}_A(S)$. Moreover, $D\mathcal{F}_A(S)$ is equivalent to the category of perfect complexes over the endomorphism algebra of the generators,*

$$D\mathcal{F}_A(S) \cong \mathrm{Perf}\left(\bigoplus_{i,j} \hom_{\mathcal{F}_A(S)}(a_i, a_j)\right).$$

The endomorphism algebra occurring in the statement above can be described as the path algebra of a quiver modulo quadratic monomial relations, as we will shortly see in an example. Before that, let us mention how $\mathcal{F}_A(S)$ relates to the partially wrapped Fukaya category.

Given a compact oriented marked surface (S, m), we can turn it into a Liouville manifold M by gluing along each component C of ∂S a cylinder with k stopping points where k is the number of components of $C \setminus m$; let σ denote the collection of stops obtained in that way. Then one can show the following equivalence.

Theorem 6.8 ([11, Thm. 9.30]). *Let (S, m) be a compact oriented marked surface and let M be a Liouville manifold with stops σ obtained from (S, m) as described above. Then there is an equivalence of A_∞-categories*

$$\mathcal{F}_A(S) \cong \mathcal{W}_\sigma(M).$$

6.3. **Homological mirror symmetry for $\mathbb{A}^1$ and $\mathbb{P}^1$.** Let us now come back to the example of $\mathbb{C}^*$ with a single stopping point. The corresponding marked surface (S, m) looks as follows:

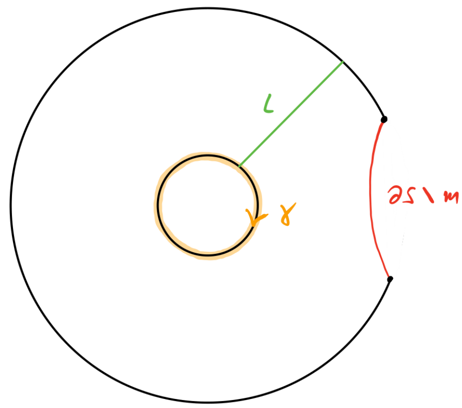


FIGURE 6.4. The marked surface (S, m) corresponding to $\mathbb{C}^*$ with one stop; the red part is $\partial S \setminus m$. Moreover, an arc L and a boundary path γ connecting L to itself

The arc L satisfies the condition of Proposition 6.7 as $S \setminus L$ is a single disk with a single component of $\partial S \setminus m$; therefore, L generates $D\mathcal{F}_{\{L\}}(S)$. The only boundary paths between L and itself are the orange shaded γ in Figure 6.3 (one loop around the inner circle) and concatenations thereof. Thus, $\hom_{\mathcal{F}_{\{L\}}(S)}(L, L)$ is generated by $\gamma, \gamma \circ \gamma, \ldots$, so $\hom_{\mathcal{F}_{\{L\}}(S)}(L, L) \cong \mathbb{C}[x]$ and therefore

$$D\mathcal{F}_{\{L\}}(S) \cong D\mathcal{W}_\sigma(T^*S^1) \cong \mathrm{Perf}(\mathbb{C}[x]) \cong D^b\mathrm{Coh}(\mathbb{A}^1).$$

Note that $\hom_{\mathcal{F}_{\{L\}}(S)}(L, L)$ is also isomorphic to the path algebra $\mathbb{C}Q$ of the quiver Q given by

$$\begin{tikzcd} L \arrow[loop above, "\gamma"] \end{tikzcd}$$

Hence the mirror dual to $\mathbb{A}^1$ is indeed the cylinder with one stopping point.

Let us now consider a slightly more complicated marked surface:

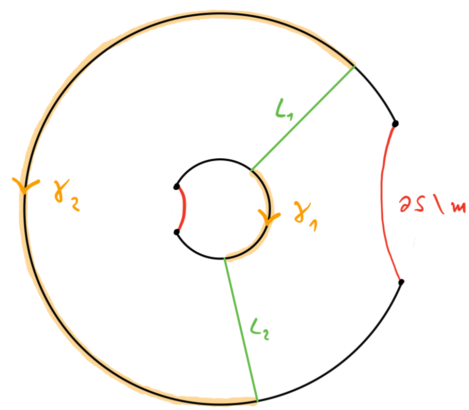


FIGURE 6.5. A marked surface (S, m); again, the red part depicts $\partial S \setminus m$, consisting of two components; moreover, two arcs L_1 and L_2 and boundary paths γ_1 and γ_2 connecting them

This time, the condition of Proposition 6.7 is satisfied by $A = \{L_1, L_2\}$. There are two boundary paths between L_1 and L_2, denoted by γ_1 and γ_2; these are the only boundary paths between them, any other path along ∂S between the L_i would not entirely lie in m, i.e. it had to use a red part. The corresponding quiver Q is

$$\begin{tikzcd} L_1 \arrow[r, bend left, "\gamma_1"] \arrow[r, bend right, "\gamma_2"'] & L_2 \end{tikzcd}$$

This is the famous Kronecker quiver. The following is a well-known result.

Theorem 6.9 (cf. [9]). *Let Q be the Kronecker quiver and let $\mathbb{C}Q$ be its path-algebra. Then there is an equivalence of triangulated categories*

$$\mathrm{Perf}(\mathbb{C}Q) \cong D^b\mathrm{Coh}(\mathbb{P}^1).$$

Let σ be the set of the two stops corresponding to the two components of $\partial S \setminus m$ in T^*S^1. Then we get an equivalence of triangulated categories

$$D\mathcal{W}_\sigma(T^*S^1) \cong D^b\mathrm{Coh}(\mathbb{P}^1).$$

Hence we identified the mirror dual to $\mathbb{P}^1$ to be the cylinder with two stopping points and proved half of the homological mirror symmetry statement.

6.4. **Conclusion.**

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