STABLE POINTED CURVES AND STABLE MAPS

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Abstract. The goal of this talk is to introduce Gromov–Witten invariants. To do so, we will introduce stable pointed curves, stable maps and their moduli spaces. In the end we will list important properties of GW invariants. The talk follows §2.1 in [1].

1. Stable pointed curves

The idea of Gromov–Witten (GW) invariants is to count curves of a given type inside a manifold which intersect a given set of submanifolds. It turns out that the "correct" notion of curve in this context is the following.

Definition 1.1. A *stable n-pointed curve* is a tuple $(C, x_1, \dots, x_n)$ such that

- C is a (possibly reducible) proper, reduced, connected, algebraic curve over an algebraically closed field with at most nodal singularities;
- $x_1, \dots, x_n \in C$ are pairwise distinct points not coinciding with any of the nodes;
- the automorphism group $\operatorname{Aut}(C, x_1, \dots, x_n)$ is finite.

Here, an *isomorphism* of n-pointed curves $(C, x_1, \dots, x_n)$ and $(C', x_1', \dots, x_n')$ is an isomorphism $\varphi\colon C \to C'$ with $\varphi(x_i) = x_i'$ for $i = 1, \dots, n$. By the *genus* of a stable curve we mean the arithmetic genus of C.

The third condition of having finite automorphisms is necessary to ensure that the moduli space of stable curves will be compact. Let us derive an equivalent, more geometric condition (for now let us work over $\mathbb{C}$, but I believe one can also show this in positive characteristic).

Lemma 1.2. *Let C be a smooth, projective, irreducible curve over $\mathbb{C}$. Then $\operatorname{Aut}(C, x_1, \dots, x_n)$ is finite $\Leftrightarrow 2g - 2 + n > 0$.*

Proof. There is an open embedding $\operatorname{Aut}(C) \hookrightarrow \operatorname{Hilb}(C \times C)$ by mapping an automorphim φ to its graph Γ_φ. Then we get

$$T_{\mathrm{id}} \operatorname{Aut}(C) \cong T_\Delta \operatorname{Hilb}(C \times C) \cong \operatorname{Hom}(\mathcal{I}_\Delta, \mathcal{O}_\Delta) \cong H^0(\Delta, \mathcal{N}_{\Delta/C\times C}) \cong H^0(C, \mathcal{T}_C), \qquad (\star)$$

where $\Delta = \Gamma_{\mathrm{id}}$ is the diagonal. Let ω_C be the canonical sheaf of C and let $m > 0$ be so that $\omega_C^{\otimes m}$ is very ample. Then we have an embedding $C \hookrightarrow \mathbb{P}(\Gamma(C, \omega_C^{\otimes m}))$. The action of $\operatorname{Aut}(C)$ extends to $\mathbb{P}(\Gamma(C, \omega_C^{\otimes m}))$ via

$$\varphi\colon \Gamma(C, \omega_C^{\otimes m}) \to \Gamma(C, \omega_C^{\otimes m}), \quad s \mapsto s \circ \varphi.$$

Thus there is an embedding $\iota\colon \operatorname{Aut}(C) \to \operatorname{PGL}(\Gamma(C, \omega_C^{\otimes m}))$; let G denote the closure of its image. Suppose $\operatorname{Aut}(C)$ is infinite, so G has positive dimension. Then there exists a $0 \neq v \in T_{\mathrm{id}}G$, which gives a nonzero $v \in T_{\mathrm{id}} \operatorname{Aut}(C)$ and thus a nonzero global section of $\mathcal{T}_C$ by $(\star)$. This implies $\deg(\omega_C) = 2g - 2 < 0$ and thus we have shown $g \geq 2 \Rightarrow \operatorname{Aut}(C)$ is finite.

Now suppose $g = 0$, so $C \cong \mathbb{P}^1$ and $\operatorname{Aut}(\mathbb{P}^1) = \operatorname{PGL}_2(\mathbb{C})$; explicitly

$$\operatorname{PGL}_2(\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{P}(\operatorname{Mat}(2 \times 2, \mathbb{C}))\colon ad - bc \neq 0 \right\}.$$

This group is well-known to be 3-transitive, so we need $n \geq 3$ to get finite automorphisms.

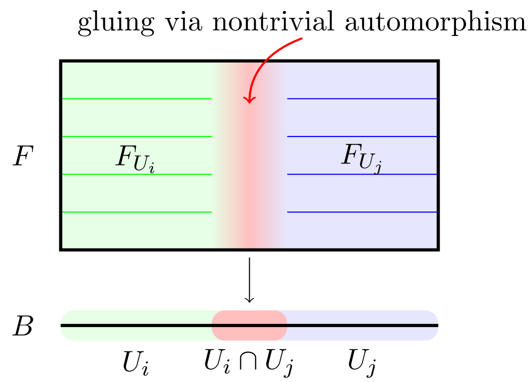

FIGURE 1. Construction of a nontrivial family F over a base scheme B via gluing along a nontrivial automorphism.

Lastly, consider the case $g = 1$, so $C = E$ is an elliptic curve. Then $\mathrm{Aut}(E) \cong E(\mathbb{C}) \rtimes G$ for $G \in \{\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/6\mathbb{Z}\}$. Moreover, it is known that $E(\mathbb{C})$ acts simply transitively on E, showing that for $n \geq 1$ the automorphism group is finite. □

Using this Lemma, we can give a geometric characterisation what it means for a pointed curve to be stable.

Proposition 1.3. *Let C be a connected, projective curve over $\mathbb{C}$ with at most nodal singularities and let $x_1, \ldots, x_n$ be n distinct smooth points. Moreover, let $\nu\colon \widetilde{C} \to C$ be the normalization of C and call a point $p \in \widetilde{C}$* distinguished *if $\nu(p)$ is either a node or one of the points $x_1, \ldots, x_n$. Then $(C, x_1, \ldots, x_n)$ is stable if and only if every irreducible component $\widetilde{C}_c \subseteq \widetilde{C}$ of the normalization satisfies either of the following conditions:*

- *$\widetilde{C}_c$ has genus 0 and contains at least 3 distinguished points; or*
- *$\widetilde{C}_c$ has genus 1 and contains at least 1 distinguished point; or*
- *$\widetilde{C}_c$ has genus at least 2.*

Proof. This is essentially a consequence from the Lemma above. Confer [3, Prop. 3.13] for details. □

Let us now consider the moduli functor between the category of schemes and the category of sets given by

$$S \mapsto \left\{ \begin{array}{c} \text{isomorphism classes of flat families } \mathcal{C} \to S \text{ with sections} \\ \sigma_1, \ldots, \sigma_n \colon S \to \mathcal{C} \text{ such that } (\mathcal{C}_{\bar{s}}, \sigma_1(\bar{s}), \ldots, \sigma_n(\bar{s})) \text{ is a stable} \\ n\text{-pointed genus } g \text{ curve for every geometric point } \bar{s} \text{ of } S \end{array} \right\}.$$

Ideally, this functor would be representable by a scheme $\overline{\mathcal{M}}_{g,n}$, constituting a fine moduli space of stable n-pointed curves of genus g, coming with a universal family. Unfortunately, this is not the case, as the existence of non-trivial automorphisms prevents the moduli functor from being representable by a scheme. A heuristic argument why this is the case is as follows: Suppose there exist non-trivial automorphisms on the families we want to parametrise. The idea is to find a base scheme B with a covering $B = \cup_i U_i$ and construct a nontrivial family F over B such that F_{U_i} is trivial for all i but the gluing

$$F_{U_i}|_{U_i \cap U_j} \cong F_{U_j}|_{U_i \cap U_j}$$

is done via some nontrivial automorphism, see Figure 1. Now suppose there is a scheme M representing the moduli functor and coming with a universal family $\mathcal{U}$. Then the map from B to M factors over the U_i and therefore the pullback of $\mathcal{U}$ is trivial, contradicting the nontriviality of F.

Instead, the moduli space $\overline{\mathcal{M}}_{g,n}$ is an *irreducible, proper, smooth Deligne–Mumford (DM) stack*. In the spirit of [1], we will not define what a stack is. One can think of a smooth DM stack as an object that locally looks like the quotient of a smooth scheme by a finite group. Many scheme-theoretic constructions are also defined for DM stacks, in particular Chow groups and intersection products are defined. But one has to be careful here: Chow groups are only defined over the *rational numbers*. Particularly, for a smooth DM stack X, the intersection product is a map $A^i_{\mathbb{Q}}(X) \times A^j_{\mathbb{Q}}(X) \to A^{i+j}_{\mathbb{Q}}(X)$ and can yield rational intersection numbers.

Example 1.4. Consider the group $G = \mathbb{Z}/2\mathbb{Z}$ acting on $\mathbb{A}^2$ via $(-1).(x, y) = (-x, -y)$. The affine quotient $\mathbb{A}^2/G$ is defined by taking Spec of the invariant ring, i.e. $\mathbb{A}^2/G = \mathrm{Spec}(\mathbb{C}[x, y]^{\mathbb{Z}/2\mathbb{Z}}) = \mathrm{Spec}(\mathbb{C}[x^2, xy, y^2])$. The stack quotient $[\mathbb{A}^2/G]$ comes with a natural map

$$[\mathbb{A}^2/G] \to \mathbb{A}^2/G$$

which is an isomorphism away from the origin. At the origin, the stack quotient $[\mathbb{A}^2/G]$ "remembers" the non-trivial stabilizer $\mathbb{Z}/2\mathbb{Z}$. Note that while $\mathbb{A}^2/G$ is singular, $[\mathbb{A}^2/G]$ is a smooth DM stack.

Similarly, consider $G = \mathbb{Z}/2\mathbb{Z}$ acting on $\mathbb{P}^2$ via $(-1).(x, y, z) = (-x, -y, z)$. Let $\pi\colon \mathbb{P}^2 \to [\mathbb{P}^2/G]$ be the projection, let L_1, L_2 denote the lines given by $x = 0$ and $y = 0$ and let D_1 and D_2 be their images in $[\mathbb{P}^2/G]$. One can define the intersection number of D_1 and D_2 at P as

$$\frac{1}{|G|} \sum_{Q \in \pi^{-1}(P)} i_Q$$

where i_Q is the intersection number of L_1 and L_2; this yields $D_1 \cdot D_2 = \frac{1}{2} L_1 \cdot L_2 = \frac{1}{2}$.

Intersection theory being defined over the rationals will also have consequences for GW invariants:

Warning 1.5. Although GW invariants are interpreted as curve counts, they are rational numbers in general.

Let us look at some examples of moduli spaces of stables curves.

Example 1.6. (1) As a consequence of Proposition 1.3, $\overline{\mathcal{M}}_{0,n}$ is empty for $n \leq 2$ and $\overline{\mathcal{M}}_{0,3}$ is a single point.

(2) Consider the space $\mathcal{M}_{0,n}$, the open subset of $\overline{\mathcal{M}}_{0,n}$ corresponding to *smooth* curves with n points, for $n \geq 3$: take a curve $(C, x_1, \ldots, x_n)$ of genus 0, then $(C, x_1, \ldots, x_n) \cong (\mathbb{P}^1, 0, 1, \infty, x'_4, \ldots, x'_n)$ via some unique map in $\mathrm{PGL}_2(\mathbb{C})$. The points $x'_4, \ldots, x'_n$ are again pairwise distinct and also distinct from $\{0, 1, \infty\}$. Therefore, we obtain

$$\mathcal{M}_{0,n} \cong (\mathbb{P}^1 \setminus \{0, 1, \infty\})^{n-3} \setminus \Delta \quad \text{for } n \geq 3$$

where $\Delta = \{(p_i)_i : \exists i \neq j \text{ such that } p_i = p_j\}$ is the big diagonal. Note that $\overline{\mathcal{M}}_{0,n}$ is always a scheme as n-pointed rational curves with $n \geq 3$ have no automorphisms.

(3) Let us look at the case $\overline{\mathcal{M}}_{0,4}$. We know it contains $\mathcal{M}_{0,4} \cong \mathbb{P}^1 \setminus \{0, 1, \infty\}$; at the boundary, the smooth $\mathbb{P}^1$ with 4 marked points can break up into two irreducible components, each containing 2 marked points and 1 nodal singularity to retain stability, in one of the three ways depicted in Figure 2. Therefore, $\overline{\mathcal{M}}_{0,4} \cong \mathbb{P}^1$, and as this is a fine moduli space it comes with a universal family obtained in the following way: consider the trivial $\mathbb{P}^1$-bundle over $\mathbb{P}^1$ with the 0-, 1- and ∞-sections, denoted $\sigma_1, \sigma_2, \sigma_3$, and the diagonal section σ_4. Then the universal family is the blow up of $\mathbb{P}^1 \times \mathbb{P}^1$ at the three points where σ_4 intersects σ_1, σ_2 or σ_3. Intuitively speaking, every time the fourth marked point x_4 approaches one of the other marked points, we have to *bubble off* a copy of $\mathbb{P}^1$.

(4) Let $\pi\colon \mathcal{C}_{0,n} \to \overline{\mathcal{M}}_{0,n}$ denote the universal family, also calles universal curve. Then $\overline{\mathcal{M}}_{0,n+1} \cong \mathcal{C}_{0,n}$, see [3, Thm. 4.22]. In particular, we have $\overline{\mathcal{M}}_{0,5} \cong \mathrm{Bl}_{(0,0),(1,1),(\infty,\infty)} \mathbb{P}^1 \times \mathbb{P}^1$.

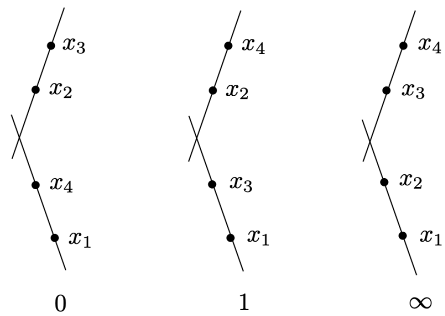

FIGURE 2. The stable singular curves in $\overline{\mathcal{M}}_{0,4}$

Actually, this holds in more generality: let $\pi_{n+1} \colon \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ be the map that forgets about the last marked point. Then π_{n+1} is the universal curve over $\overline{\mathcal{M}}_{g,n}$ (note that this is in general not a map of schemes though).

(5) The space $\overline{\mathcal{M}}_{1,1}$ is given by the j-line $\mathbb{A}^1_j$, the coarse moduli space of elliptic curves, together with some stacky information. For $j \notin \{0, 1\}$, the corresponding pointed elliptic curve has precisely one automorphism given by negation. Therefore, points $j \notin \{0, 1\}$ are locally of the form $U/\mathbb{Z}_2$, where $\mathbb{Z}_2$ acts trivially on U. For $j \in \{0, 1\}$, the automorphism groups are larger and the stacky information is more complicated.

Fact 1.7. $\dim(\overline{\mathcal{M}}_{g,n}) = 3g - 3 + n$.

2. Stable maps

We now want to pass to a setting where we consider stable curves inside an ambient space.

Definition 2.1. Let X be a variety. A *stable n-pointed map* to X is a map $f \colon (C, x_1, \dots, x_n) \to X$ such that

- C is a (possibly reducible) proper, reduced, connected, algebraic curve with at most nodal singularities;
- $x_1, \dots, x_n \in C$ are pairwise distinct points not coinciding with any of the nodes;
- f has finite automorphism group.

Here, an *isomorphism* of n-pointed maps $f \colon (C, x_1, \dots, x_n) \to X$ and $f' \colon (C', x_1', \dots, x_n') \to X$ is an isomorphism $\varphi \colon (C, x_1, \dots, x_n) \to (C', x_1', \dots, x_n')$ of n-pointed curves with $f' \circ \varphi = f$.

For a homology class $\beta \in H_2(X, \mathbb{Z})$, we say that f *represents* β if $f_\star([C]) = \beta$, where $[C] \in H_2(C, \mathbb{Z})$ is the fundamental class of C. We can then consider the following moduli functor

$$S \mapsto \left\{ \begin{array}{c} \text{isomorphism classes of flat families } \mathcal{C} \to S \text{ with sections} \\ \sigma_1, \dots, \sigma_n \colon S \to \mathcal{C} \text{ and a morphism } f \colon \mathcal{C} \to X \text{ such that} \\ f \colon (\mathcal{C}_{\bar{s}}, \sigma_1(\bar{s}), \dots, \sigma_n(\bar{s})) \to X \text{ is a stable } n\text{-pointed map} \\ \text{of genus } g \text{ representing } \beta \text{ for every geometric point } \bar{s} \text{ of } S \end{array} \right\}.$$

Again, this moduli functor is in general not representable by a scheme. Instead, there is a proper Deligne–Mumford stack, denoted $\overline{\mathcal{M}}_{g,n}(X, \beta)$ representing this functor. This time we are even a bit more unlucky than before because $\overline{\mathcal{M}}_{g,n}(X, \beta)$ is in general not smooth.

Example 2.2. Take the target space $X = \mathbb{P}^2$ and let $[l] \in H_2(X, \mathbb{Z})$ be the homology class of a line. In the following we will denote the moduli space $\overline{\mathcal{M}}_{g,n}(X, d[l])$ simply by $\overline{\mathcal{M}}_{g,n}(X, d)$.

(1) $\overline{\mathcal{M}}_{0,0}(X, 1) = (\mathbb{P}^2)^*$
(2) $\overline{\mathcal{M}}_{0,1}(X, 1) = \{(x, l) \colon x \in l\} \subseteq \mathbb{P}^2 \times (\mathbb{P}^2)^*$

(3) $\overline{\mathcal{M}}_{0,0}(X,2)$ parametrises four different types of stable maps:
   (a) C may be irreducible and $f(C)$ is a conic
   (b) C may be a union of two lines and $f(C)$ a reducible conic
   (c) C may be irreducible and $f(C)$ is a double cover of a line
   (d) C may be a union of two lines and $f(C)$ is a double cover of a line

   Stable maps of type (c) and (d) give stacky points as these have non-trivial automorphisms.

(4) Consider $\overline{\mathcal{M}}_{1,0}(X,1)$: although there are no maps from an elliptic curve to $\mathbb{P}^2$ representing $[l]$, the domain can be reducible with $C = C_1 \cup C_2$ where C_1 is a $\mathbb{P}^1$ which gets mapped to l and C_2 is an elliptic curve or a nodal rational curve attached to C_1 at one point; C_2 gets mapped to a constant. Therefore,
$$\overline{\mathcal{M}}_{1,0}(X,1) = \overline{\mathcal{M}}_{0,1}(X,1) \times \overline{\mathcal{M}}_{1,1},$$
   where the marked point corresponds to the point where C_1 and C_2 meet.

Ideally, we would like to define GW invariants by integration against the fundamental class of $\overline{\mathcal{M}}_{g,n}(X,\beta)$. Unfortunately, the moduli space might not have the "correct" dimension to yield a meaningful counting. To remedy this problem, one constructs a *virtual fundamental class*
$$[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir}} \in A_d(\overline{\mathcal{M}}_{g,n}(X,\beta)) \otimes \mathbb{Q}$$
where d is the *expected* (or *virtual*) *dimension* of $\overline{\mathcal{M}}_{g,n}(X,\beta)$. The construction of virtual fundamental classes is quite involved and we will not describe it here, see [4, §4] for an exposition. The idea is that locally the moduli space is cut out by equations coming from an obstruction space inside a space coming from first order deformations. Let $[f\colon (C,x_1,\dots,x_n)\to X] \in \overline{\mathcal{M}}_{g,n}(X,\beta)$ be a stable map and let $\boldsymbol{I}^\bullet$ be the 2-term complex $\boldsymbol{I}^\bullet := f^*\Omega_X \to \Omega^1_C(\sum_i x_i) \in D^b(C)$, concentrated in degrees -1 and 0. Then the infinitesimal deformations are given by
$$T_{[f]}\overline{\mathcal{M}}_{g,n}(X,\beta) = \mathrm{Ext}^1_C(I^\bullet, \mathcal{O}_C)$$
while the obstruction space is given by $\mathrm{Ext}^2_C(I^\bullet,\mathcal{O}_C)$. Using Hirzebruch–Riemann–Roch Theorem, one can then show that
$$\begin{aligned}\mathrm{vdim}(\overline{\mathcal{M}}_{g,n}(X,\beta)) &= \dim(\mathrm{Ext}^1_C(I^\bullet,\mathcal{O}_C)) - \dim(\mathrm{Ext}^2_C(I^\bullet,\mathcal{O}_C))\\ &= n + (\dim_{\mathbb{C}} X - 3)(1-g) + \int_\beta c_1(\mathcal{T}_X). \end{aligned} \qquad (\spadesuit)$$
Note that if X is a Calabi–Yau 3-fold, $\mathrm{vdim}(\overline{\mathcal{M}}_{g,0}(X,\beta)) = 0$.

3. Gromov–Witten invariants

Given the moduli space of n-pointed stable maps of genus g representing β, there exist evaluation maps
$$\mathrm{ev}_i\colon \overline{\mathcal{M}}_{g,n}(X,\beta) \to X \quad \text{for } i = 1,\dots,n$$
evaluating a stable map at the i^{th} marked point, i.e.
$$\mathrm{ev}_i\colon [f\colon (C,x_1,\dots,x_n)\to X] \mapsto f(x_i).$$
We also define
$$\mathrm{ev} := \mathrm{ev}_1 \times \cdots \times \mathrm{ev}_n\colon \overline{\mathcal{M}}_{g,n}(X,\beta) \to X^n.$$

Definition 3.1. For cohomology classes $\alpha_1,\dots,\alpha_n \in H^*(X,\mathbb{Q})$ and $\beta \in H_2(X,\mathbb{Z})$, we define the *Gromov–Witten (GW) invariant*
$$\langle \alpha_1,\dots,\alpha_n\rangle_{g,\beta} := \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir}}} \mathrm{ev}^*(\alpha_1\times\cdots\times\alpha_n) \in \mathbb{Q}.$$

As usual, in this context the integral means taking the degree of the cap product. This integral is only nonzero if the degrees of the homology and the cohomology class match; in this case, this means that if $\alpha_i \in H^{d_i}(X,\mathbb{Q})$ then $\langle \alpha_1, \dots, \alpha_n \rangle_{g,\beta} \neq 0$ if and only if $2\,\mathrm{vdim}(\overline{\mathcal{M}}_{g,n}(X,\beta)) = \sum_{i=1}^n d_i$, i.e. by (♠)

$$2\left(n + (\dim_{\mathbb{C}} X - 3)(1-g) + \int_\beta c_1(\mathcal{T}_X)\right) = \sum_{i=1}^n d_i. \tag{$\diamondsuit$}$$

Coming back to the very first sentence of this talk, the intuition behind GW invariants should be that they count the number of curves of homology class β and genus g inside X passing through $Y_1, \dots, Y_n$, where Y_i is the submanifold Poincaré dual to α_i.

Example 3.2. Let us look at the case $X = \mathbb{P}^2$ and genus 0; then $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^2, d)$ is actually a smooth DM stack for $d \geq 0$ (and $n \geq 3$ for $d = 0$) of the expected dimension. If $\alpha_1, \dots, \alpha_n \in H^4(X,\mathbb{Q})$ are taken to be the Poincaré dual of a point [pt], it can be shown that the GW invariants actually satisfy the intuition stated above, i.e. $\langle \alpha_1, \dots, \alpha_n \rangle_{0,d[l]}$ coincides with the number of rational curves of degree d passing through n general points in $\mathbb{P}^2$ as long as this number is expected to be finite. Equation ($\diamondsuit$) tells us that this is precisely the case if $2(n-1+3d) = 4n$, i.e. $n = 3d-1$.

(1) $\langle [\mathrm{pt}], [\mathrm{pt}] \rangle_{0,[l]} = 1$ since there is precisely one line passing through two general points.
(2) $\langle [\mathrm{pt}]^5 \rangle_{0,2[l]} = 1$ since there is a unique conic passing through five general points in $\mathbb{P}^2$.
(3) The next GW number we want to compute is $\langle [\mathrm{pt}]^8 \rangle_{0,3[l]}$. A cubic curve in $\mathbb{P}^2$ is given by the vanishing of a homogeneous polynomial of degree three

$$f = c_0x^3 + c_1x^2y + c_2x^2z + \cdots + c_9z^3.$$

For generic coefficients, $\mathcal{V}(f)$ is smooth of genus one and is rational if and only if it is singular. The locus where $\mathcal{V}(f)$ becomes singular is given by the vanishing of the discriminant $\Delta(f)$, where

$$\Delta(f) = 19683c_0^4c_6^4c_9^4 - 26244c_0^4c_6^3c_7c_8c_9^3 + \cdots - c_2^2c_3c_4^4c_5^3c_6^2$$

is a homogeneous polynomial of degree 12. Passing through a point in $\mathbb{P}^2$ imposes a linear condition on the coefficients c_i, and therefore $\langle [\mathrm{pt}]^8 \rangle_{0,3[l]} = 12$.

We will now list some important properties of GW invariants.

(1) **Fundamental Class Axiom.** If $n + 2g \geq 4$ or $\beta \neq 0$ and $n \geq 1$, and $[X] \in H^0(X,\mathbb{Q})$ is the fundamental class of X, then

$$\langle \alpha_1, \dots, \alpha_{n-1}, [X] \rangle_{g,\beta} = 0.$$

(2) **Divisor Axiom.** If $n + 2g \geq 4$ or $\beta \neq 0$ and $n \geq 1$, and $\alpha_n \in H^2(X,\mathbb{Q})$, then

$$\langle \alpha_1, \dots, \alpha_n \rangle_{g,\beta} = \left(\int_\beta \alpha_n\right) \langle \alpha_1, \dots, \alpha_{n-1} \rangle_{g,\beta}.$$

(3) **Point Mapping Axiom.** For $g = 0, \beta = 0$,

$$\langle \alpha_1, \dots, \alpha_n \rangle_{0,0} = \begin{cases} \int_X \alpha_1 \cup \alpha_2 \cup \alpha_3 & \text{if } n = 3, \\ 0 & \text{otherwise.} \end{cases}$$

Note that this is not a complete list to define GW invariants in an axiomatic way, see [2].

4. Descendent Gromov–Witten invariants

In the course of the reading group we will also need a slightly different version of GW invariants, called *descendent* GW invariants, or *gravitational descendent invariants.* On $\overline{\mathcal{M}}_{g,n}(X,\beta)$ there are line bundles $\mathcal{L}_i$ for $i = 1,\dots,n$ whose fiber at a stable curve $[(C,x_1,\dots,x_n)]$ is the cotangent space $\mathfrak{m}_{x_i}/\mathfrak{m}_{x_i}^2$ where $\mathfrak{m}_{x_i} \subseteq \mathcal{O}_{C,x_i}$ is the maximal ideal at x_i. More precisely, let $\sigma_i\colon \overline{\mathcal{M}}_{g,n} \to \mathcal{C}_{g,n}$ denote the section of the universal curve corresponding to x_i; then $\mathcal{L}_i = \sigma_i^*\omega_\pi$ where ω_π is the relative dualizing sheaf of the universal curve $\pi\colon \mathcal{C}_{g,n} \to \overline{\mathcal{M}}_{g,n}$. The *psi classes* ψ_i are defined as $\psi_i = c_1(\mathcal{L}_1)$.

Definition 4.1. For classes $\alpha_1,\dots,\alpha_n \in H^*(X,\mathbb{Q}), \beta \in H_2(X,\mathbb{Z})$ and non-negative integers $p_1,\dots,p_n \in \mathbb{Z}_{\geq 0}$, the *descendent Gromov–Witten invariant* is defined by

$$\langle \psi^{p_1}\alpha_1,\dots,\psi^{p_n}\alpha_n\rangle_{g,\beta} \coloneqq \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir}}} \psi_1^{p_1} \cup \dots \cup \psi_n^{p_n} \cup \mathrm{ev}^*(\alpha_1 \times \dots \times \alpha_n) \in \mathbb{Q}.$$

Similar to ($\diamondsuit$), this expression is only non-zero if

$$2\left(n + (\dim_{\mathbb{C}} X - 3)(1-g) + \int_\beta c_1(\mathcal{T}_X)\right) = \sum_{i=1}^n d_i + \sum_{i=1}^n 2p_i.$$

Example 4.2. Even for target space $X = \mathbb{P}^2$ and genus 0, descendent GW invariants can take non-integral values, e.g. $\langle \psi^4[pt]\rangle_{0,2[l]} = \frac{1}{8}$. This number will probably be computed in a later talk using tropical geometry.

There exist generalisations of the properties listed in the previous section to descendent invariants.

(1) **Fundamental Class Axiom.** If $n + 2g \geq 4$ or $\beta \neq 0$ and $n \geq 1$, and $[X] \in H^0(X,\mathbb{Q})$ is the fundamental class of X, then

$$\langle \psi^{p_1}\alpha_1,\dots,\psi^{p_{n-1}}\alpha_{n-1},[X]\rangle_{g,\beta} = \sum_{i=1}^{n-1} \langle \psi^{p_1}\alpha_1,\dots,\psi^{p_i-1}\alpha_i,\dots,\psi^{p_{n-1}}\alpha_{n-1}\rangle_{g,\beta}$$

where we use the convention that an invariant on the RHS is zero if ψ appears with a negative power.

(2) **Divisor Axiom.** If $n + 2g \geq 4$ or $\beta \neq 0$ and $n \geq 1$, and $\alpha_n \in H^2(X,\mathbb{Q})$, then

$$\begin{aligned}\langle \psi^{p_1}\alpha_1,\dots,\psi^{p_{n-1}}\alpha_{n-1},\alpha_n\rangle_{g,\beta} = {} & \left(\int_\beta \alpha_n\right) \langle \psi^{p_1}\alpha_1,\dots,\psi^{p_{n-1}}\alpha_{n-1}\rangle_{g,\beta} \\ & + \sum_{i=1}^{n-1} \langle \psi^{p_1}\alpha_1,\dots,\psi^{p_i-1}(\alpha_n \cup \alpha_i),\dots,\psi^{p_{n-1}}\alpha_{n-1}\rangle_{g,\beta},\end{aligned}$$

where we use the same convention as above.

(3) **Point Mapping Axiom.** For $g = 0, \beta = 0$ and $n \leq 3$,

$$\langle \psi^{\nu_1}\alpha_1,\dots,\psi^{\nu_n}\alpha_n\rangle_{0,0} = \begin{cases} \int_X \alpha_1 \cup \alpha_2 \cup \alpha_3 & \text{if } n = 3 \text{ and } \nu_1 = \dots = \nu_n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

There is another important property we only see in the descendent case:

(4) **Dilaton Axiom.**

$$\langle \psi[X],\psi^{p_1}\alpha_1,\dots,\psi^{p_n}\alpha_n\rangle_{g,\beta} = (2g-2+n)\langle \psi^{p_1}\alpha_1,\dots,\psi^{p_n}\alpha_n\rangle_{g,\beta}.$$

References

[1] M. Gross. Tropical geometry and mirror symmetry. Number 114 in CBMS Regional Conference Series in Mathematics. AMS, 2011.

[2] M. Kontsevich and Y. Manin. Gromov-witten classes, quantum cohomology, and enumerative geometry. *Communications in Mathematical Physics*, 164:525–562, 1994.

[3] J. Schmitt. The moduli space of curves, September 2020. Lecture notes from the University of Bonn.

[4] M. Wiesmann. Stable pair invariants after Pandharipande–Thomas, July 2021.